OPERATOR FORMALISM AND FOCK SPACE

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Reference: Sidney Coleman, *Quantum Field Theory: Lectures of Sidney Coleman* (edited by Bryan Gin-ge Chen *et al.*), World Scientific, 2019. Section 2.4.

Post date: 13 Dec 2019.

In Section 2.4, Coleman applies the raising and lowering operators of the harmonic oscillator to the multi-particle system of Fock space. The raising and lowering operators $a^{\dagger}$ and a are reinterpreted as creation and annihilation operators which create and annihilate particles in Fock space states.

To summarize his results:

The operator $a_{\boldsymbol{p}}^{\dagger}$ creates a particle with 3-momentum $\boldsymbol{p}$ and $a_{\boldsymbol{p}}$ annihilates a particle with momentum $\boldsymbol{p}$. The operators satisfy the commutation relations

$$\left[a_{\boldsymbol{p}},a_{\boldsymbol{p}'}\right]=\left[a_{\boldsymbol{p}}^{\dagger},a_{\boldsymbol{p}'}^{\dagger}\right]=0 \tag{1}$$

$$\left[a_{\boldsymbol{p}},a_{\boldsymbol{p}'}^{\dagger}\right]=\delta^{(3)}\left(\boldsymbol{p}-\boldsymbol{p}'\right) \tag{2}$$

The number operator counts the number of particles with momentum $\boldsymbol{p}$ and is given by

$$N\left(\boldsymbol{p}\right)=a_{\boldsymbol{p}}^{\dagger}a_{\boldsymbol{p}} \tag{3}$$

The Hamiltonian and total momentum operators are given by

$$H=\int d^{3}\boldsymbol{p}\,\omega_{\boldsymbol{p}}N\left(\boldsymbol{p}\right) \tag{4}$$

$$\boldsymbol{P}=\int d^{3}\boldsymbol{p}\,\boldsymbol{p}N\left(\boldsymbol{p}\right) \tag{5}$$

where $\omega_{\boldsymbol{p}}$ is the energy of a particle with momentum $\boldsymbol{p}$.

The single particle state with momentum $\boldsymbol{p}$ is written $\left|\boldsymbol{p}\right\rangle$ and is created from the vacuum state by

$$\left|\boldsymbol{p}\right\rangle=a_{\boldsymbol{p}}^{\dagger}\left|0\right\rangle \tag{6}$$

and has normalizaton

$$\left\langle \boldsymbol{p}' | \boldsymbol{p} \right\rangle = \delta^{(3)}\left(\boldsymbol{p} - \boldsymbol{p}'\right) \tag{7}$$

Relativistic normalization of creation and annihilation operators. We defined relativistic one-particle states by

$$|p\rangle = \sqrt{(2\pi)^3}\sqrt{2\omega_{\boldsymbol{p}}}\,|\boldsymbol{p}\rangle \tag{8}$$

where p is a 4-momentum.

We can define a creation operator $\alpha^{\dagger}(p)$ that creates a particle in a 4-momentum state by

$$\alpha^{\dagger}(p) = (2\pi)^{3/2}\sqrt{2\omega_{\boldsymbol{p}}}a_{\boldsymbol{p}}^{\dagger} \tag{9}$$

The related annihilation operator is then

$$\alpha(p) = (2\pi)^{3/2}\sqrt{2\omega_{\boldsymbol{p}}}a_{\boldsymbol{p}} \tag{10}$$

Coleman then shows that the Lorentz transformation of these operators is given by

$$\alpha^{\dagger}(\Lambda p) = U(\Lambda)\,\alpha^{\dagger}(p)\,U^{\dagger}(\Lambda) \tag{11}$$

$$\alpha(\Lambda p) = U(\Lambda)\,\alpha^{\dagger}(p)\,U(\Lambda) \tag{12}$$

where $U(\Lambda)$ is the unitary operator that provides a Lorentz transformation Λ, and satisfies

$$U(\Lambda)|0\rangle = |0\rangle \tag{13}$$

$$U(\Lambda)|p\rangle = |\Lambda p\rangle \tag{14}$$

That is, the vacuum state is invariant under a Lorentz transformation, and the Lorentz transformation of a state with 4-momentum p produces a state with 4-momentum Λp.

We can apply the same logic to the Lorentz transformation of a multi-particle state, in which all the particles undergo the same transformation. We get

$$U(\Lambda)|p_1, p_2, \ldots, p_n\rangle = |\Lambda p_1, \Lambda p_2, \ldots, \Lambda p_n\rangle \tag{15}$$

For translation by a fixed 4-vector a, the unitary operator is

$$U(a) = e^{iP\cdot a} \tag{16}$$

where P is the 4-momentum operator. We assume the vacuum is unchanged under translation so that

$$U(a)|0\rangle = |0\rangle \tag{17}$$

A multi-particle state transforms according to

$$U(a)|p_1, p_2, \ldots, p_n\rangle = e^{ia\cdot\Sigma p_i}|p_1, p_2, \ldots, p_n\rangle \tag{18}$$

since the operator P acts on all the particles and adds up the momentum of each one to produce the total momentum $\sum p_i$.

We can work out the translation property of $\alpha^\dagger(p)$ as follows. Consider a single particle state $|p\rangle$. Then

$$U(a)|p\rangle = U(a)\alpha^\dagger(p)|0\rangle \tag{19}$$
$$= U(a)\alpha^\dagger(p)U^\dagger(a)U(a)|0\rangle \tag{20}$$
$$= U(a)\alpha^\dagger(p)U^\dagger(a)|0\rangle \tag{21}$$
$$= e^{ip\cdot a}|p\rangle \tag{22}$$
$$= e^{ip\cdot a}\alpha^\dagger(p)|0\rangle \tag{23}$$

We used 17 to go from the second to the third line, and 16 to get the fourth line from the LHS of the first line. If we compare the third and last lines, we have

$$U(a)\alpha^\dagger(p)U^\dagger(a) = e^{ip\cdot a}\alpha^\dagger(p) \tag{24}$$

or

$$e^{iP\cdot a}\alpha^\dagger(p)e^{-iP\cdot a} = e^{ip\cdot a}\alpha^\dagger(p) \tag{25}$$

Taking the adjoint gives the relation for $\alpha(p)$:

$$e^{iP\cdot a}\alpha(p)e^{-iP\cdot a} = e^{-ip\cdot a}\alpha(p) \tag{26}$$

PINGBACKS

Pingback: Constructing a scalar quantum field