CONSTRUCTING A SCALAR QUANTUM FIELD

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Reference: Sidney Coleman, *Quantum Field Theory: Lectures of Sidney Coleman* (edited by Bryan Gin-ge Chen *et al.*), World Scientific, 2019. Sections 3.1 - 3.3.

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Chapter 3 begins with a discussion of how and why we arrive at the notion of a quantum field. The discussion in the book is quite comprehensive so I'll just summarize the key points here for future reference.

The key point is that we want a quantum theory that is consistent with special relativity. That is, any two spacetime points that have a spacelike separation cannot influence each other, so any operators that represent observable quantities (that is, hermitian operators) that depend on spacetime must commute with each other if they are separated by a spacelike interval.

In classical physics, the theory of electromagnetism as portrayed in Maxwell's equations *does* satisfy the constraints of special relativity. A change in the charge configuration at one spacetime point can propagate out from the point no faster than the speed of light, so any two events separated by a spacelike interval cannot affect each other. Classical electromagnetism doesn't use operators to represent observables, of course, but at least the principle of relativity is satisfied.

At the heart of Maxwell's equations are the electric and magnetic fields. The idea behind quantum field theory (at least, approached from this angle) is to carry over the concept of a classical field to quantum theory, and postulate the existence of a quantum field theory in which observables are represented by operators that satisfy the commutation restriction described above: any two observable operators separated by a spacelike interval must commute.

We therefore propose a set of N quantum fields $\phi^{a}(x)$ with $a=1,\ldots,N$ that depend on the spacetime point x. Coleman then gives five conditions that a scalar quantum field should satisfy.

(1) $\left[\phi^{a}(x),\phi^{b}(y)\right]=0$ if $(x-y)^{2}<0$. Recall that the square of a spacetime separation is negative for spacelike separations. This condition ensures that no signals can travel faster than light.

(2) $\phi^{a}(x)=\phi^{a}(x)^{\dagger}$. The fields are hermitian operators and are therefore observable.

(3) $e^{-iP\cdot y}\phi^a(x)e^{iP\cdot y}=\phi^a(x-y)$. Fields transform properly under translation. In this case, the translation is by a distance $+y$, which means that the field at location x after translation will be the same as the field before translation at position $x-y$. If we're dealing with one dimension, the translation is equivalent to moving the field a distance y to the right.

(4) $U(\Lambda)^\dagger\phi^a(x)U(\Lambda)=\phi^a\left(\Lambda^{-1}x\right)$, where Λ is a Lorentz transformation. The Λ^{-1} appears for the same reason as the $-y$ in condition 3: if we Lorentz transform a function by Λ, the function will have the value that it had at the original configuration Λ^{-1}.

(5) The fields are linear combinations of the creation and annihilation operators, so that

$$\phi^a(x)=\int d^3\boldsymbol{p}\left[F^a_{\boldsymbol{p}}(x)a_{\boldsymbol{p}}+G^a_{\boldsymbol{p}}(x)a^\dagger_{\boldsymbol{p}}\right] \tag{1}$$

Conditions 1 and 2 are required by the physics of the situation. We must satisfy relativistic causality, and the fields must be observable. The other conditions are imposed only for convenience at this stage. If we can build a theory satisfying conditions 3, 4 and 5, then we will have a simpler theory than if we need to complicate things by, for example, requiring that the fields are non-linear combinations of a and $a^\dagger$. As it turns out, we can build a quantum field theory satisfying all five conditions.

In section 3.3, Coleman constructs an explicit form of a scalar quantum field. He uses the relativistic creation and annihilation operators defined by

$$\alpha^\dagger(p)=(2\pi)^{3/2}\sqrt{2\omega_{\boldsymbol{p}}}a^\dagger_{\boldsymbol{p}} \tag{2}$$

$$\alpha(p)=(2\pi)^{3/2}\sqrt{2\omega_{\boldsymbol{p}}}a_{\boldsymbol{p}} \tag{3}$$

We can then write 1 for a single field at $x=0$ as

$$\phi(0)=\int\frac{d^3\boldsymbol{p}`}{(2\pi)^3(2\omega_{\boldsymbol{p}})}\left[f_p\alpha(p)+g_p\alpha^\dagger(p)\right] \tag{4}$$

where f_p and g_p are some functions of the four-momentum p, but with the usual constraint that $p^0=\omega_{\boldsymbol{p}}=\sqrt{\boldsymbol{p}^2+\mu^2}$, for a particle of mass μ.

He then shows that if we apply condition 4 (the Lorentz transformation) above and use the result

$$\alpha^\dagger(\Lambda p)=U(\Lambda)\alpha^\dagger(p)U^\dagger(\Lambda) \tag{5}$$

$$\alpha(\Lambda p)=U(\Lambda)\alpha^\dagger(p)U(\Lambda) \tag{6}$$

we find that

$$\phi(0)=\int\frac{d^{3}\boldsymbol{p}}{(2\pi)^{3}(2\omega_{\boldsymbol{p}})}\left[f_{p}\alpha(\Lambda p)+g_{p}\alpha^{\dagger}(\Lambda p)\right]\tag{7}$$

In doing this derivation, he assumes that the operator $U(\Lambda)$ has no effect on the functions f_p and g_p. It would seem to me that in order for this to be true, f_p and g_p need to be Lorentz invariants, or in other words, functions of p^2 rather than just p.

At any rate, if we now do a change of variable in 7 so that

$$p'=\Lambda p\tag{8}$$

and use the fact that the integral is over the Lorentz invariant measure $d^3\boldsymbol{p}/2\omega_{\boldsymbol{p}}$, we can write 7 as

$$\phi(0)=\int\frac{d^{3}\boldsymbol{p}}{(2\pi)^{3}(2\omega_{\boldsymbol{p}})}\left[f_{\Lambda^{-1}p}\alpha(p)+g_{\Lambda^{-1}p}\alpha^{\dagger}(p)\right]\tag{9}$$

This must be the same as 4 and, since the Lorentz transformation Λ can be anything we like (providing it is a continuous transformation), we must have

$$f_{\Lambda^{-1}p}=f_{p}\tag{10}$$

$$g_{\Lambda^{-1}p}=g_{p}\tag{11}$$

for all p and all Λ. In other words, f and g must be constants, independent of p and Λ.

Thus we have, for $x=0$:

$$\phi(0)=\int\frac{d^{3}\boldsymbol{p}}{(2\pi)^{3}(2\omega_{\boldsymbol{p}})}\left[f\alpha(p)+g\alpha^{\dagger}(p)\right]\tag{12}$$

We can now apply condition 3 above to generate the value of $\phi(x)$ at any other point x from $\phi(0)$. The result is Coleman's equation 3.31:

$$\phi(x)=\int\frac{d^{3}\boldsymbol{p}}{(2\pi)^{3}(2\omega_{\boldsymbol{p}})}\left[fe^{-ip\cdot x}\alpha(p)+ge^{ip\cdot x}\alpha^{\dagger}(p)\right]\tag{13}$$

We can verify directly that a field of this form satisfies conditions 3, 4 and 5. It's a linear function of α and $\alpha^\dagger$ so condition 5 is satisfied.

For condition 3, we have

$$e^{iP\cdot y}\phi(x)e^{-iP\cdot y}=\int\frac{d^{3}\boldsymbol{p}}{(2\pi)^{3}(2\omega_{\boldsymbol{p}})}\left[fe^{-ip\cdot x}e^{iP\cdot y}\alpha(p)e^{-iP\cdot y}+ge^{ip\cdot x}e^{iP\cdot y}\alpha^{\dagger}(p)e^{-iP\cdot y}\right]\tag{14}$$

We can use the results from earlier for the translation property of the operators:

$$e^{iP\cdot a}\alpha^{\dagger}(p)\,e^{-iP\cdot a} = e^{ip\cdot a}\alpha^{\dagger}$$
$$e^{iP\cdot a}\alpha(p)\,e^{-iP\cdot a} = e^{-ip\cdot a}\alpha(p)$$

This gives:

$$e^{-iP\cdot y}\phi(x)\,e^{iP\cdot y} = \int\frac{d^3\boldsymbol{p}}{(2\pi)^3(2\omega_{\boldsymbol{p}})}\left[fe^{-ip\cdot x}e^{ip\cdot y}\alpha(p)+ge^{ip\cdot x}e^{-ip\cdot y}\alpha^{\dagger}(p)\right] \tag{15}$$
$$= \int\frac{d^3\boldsymbol{p}}{(2\pi)^3(2\omega_{\boldsymbol{p}})}\left[fe^{-ip\cdot(x-y)}\alpha(p)+ge^{ip\cdot(x-y)}\alpha^{\dagger}(p)\right] \tag{16}$$
$$= \phi(x-y) \tag{17}$$

A similar calculation works for condition 4. We have, using 2

$$U(\Lambda)^{\dagger}\phi(x)\,U(\Lambda) = \int\frac{d^3\boldsymbol{p}}{(2\pi)^3(2\omega_{\boldsymbol{p}})}\left[fe^{-ip\cdot x}U(\Lambda)^{\dagger}\alpha(p)\,U(\Lambda)+ge^{ip\cdot x}U(\Lambda)^{\dagger}\alpha^{\dagger}(p)\,U(\Lambda)\right]$$
$$= \int\frac{d^3\boldsymbol{p}}{(2\pi)^3(2\omega_{\boldsymbol{p}})}\left[fe^{-ip\cdot x}\alpha\left(\Lambda^{-1}p\right)+ge^{ip\cdot x}\alpha^{\dagger}\left(\Lambda^{-1}p\right)\right]$$
$$= \phi\left(\Lambda^{-1}p\right)$$

To satisfy conditions 1 and 2, Coleman proposes two independent versions for ϕ in his equation 3.34. These use the partial fields

Note that we've reverted to using a instead of α for the operators.

$$\phi^{(+)}(x) = \int\frac{d^3\boldsymbol{p}}{\sqrt{(2\pi)^3}\sqrt{2\omega_{\boldsymbol{p}}}}a_{\boldsymbol{p}}e^{-ip\cdot x} \tag{18}$$
$$\phi^{(-)}(x) = \int\frac{d^3\boldsymbol{p}}{\sqrt{(2\pi)^3}\sqrt{2\omega_{\boldsymbol{p}}}}a_{\boldsymbol{p}}^{\dagger}e^{ip\cdot x} \tag{19}$$

The two independent forms are

$$\phi^1(x) = \phi^{(+)}(x)+\phi^{(-)}(x) \tag{20}$$
$$\phi^2(x) = i\left(\phi^{(+)}(x)-\phi^{(-)}(x)\right) \tag{21}$$

Coleman then works out the commutator in condition 1 for both versions of ϕ^a and shows that, for ϕ^1, the commutator is not zero for spacelike separations, but rather comes out to

$$\left[\phi^1(x),\phi^2(y)\right] = -i\Delta_+(x-y) - i\Delta_+(y-x) \tag{22}$$

where

$$\Delta_+(x-y) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3(2\omega_{\boldsymbol{p}})} e^{-ip\cdot(x-y)} \tag{23}$$

The time derivative of Δ_+ comes out to

$$\frac{\partial}{\partial x^0}\Delta_+(x) = -\frac{i}{2}\int \frac{d^3\boldsymbol{p}}{(2\pi)^3} e^{-ip\cdot x} \tag{24}$$

which is the same integral we looked at earlier using contour integration, and determined that it isn't zero. If a function has a non-zero derivative, then the function itself cannot be identically zero, so using the two independent forms 20 doesn't give us a field that satisfies causality.

The only correct solution is to use ϕ^1 on its own, and we find (Coleman equation 3.42) that (calling ϕ^1 just ϕ):

$$[\phi(x),\phi(y)] = \Delta_+(x-y) - \Delta_+(y-x) \equiv i\Delta(x-y) \tag{25}$$

This *does* turn out to be zero for spacelike intervals, according to the following argument. First, we observe that Δ_+, and therefore Δ, is Lorentz invariant, since from its definition 23, it's the integral of the Lorentz invariant measure $d^3\boldsymbol{p}/2\omega_{\boldsymbol{p}}$ multiplied by the exponential of a Lorentz scalar $p\cdot(x-y)$. A Lorentz invariant quantity can be worked out in any inertial frame. A spacelike interval is one where the two events must always be separated by a non-zero spatial distance, no matter what inertial frame you use. However, the two events can occur in either order (or simultaneously) with regard to time. As shown in footnote 8 in the book, any spacelike interval can be turned into its negative by a proper Lorentz transformation, so $x-y$ and $y-x$ are related by a single proper Lorentz transformation. Thus $\Delta_+(x-y) = \Delta_+(y-x)$ and thus $\Delta(x-y) = 0$.

The same argument doesn't work for two spacetime points with a timelike separation, since there is always a non-zero time (of the same sign) between these two events. In other words, one event can be connected to the other by a light signal, so we can't reverse the temporal order of the events, meaning that we can't transform $x-y$ to $y-x$ by a single Lorentz transformation.

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