EXTENSION OF CLASSICAL FIELD THEORY TO QUANTUM FIELD THEORY

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Reference: Sidney Coleman, *Quantum Field Theory: Lectures of Sidney Coleman* (edited by Bryan Gin-ge Chen *et al.*), World Scientific, 2019. Sections 4.4-4.5.

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Coleman extends classical field theory to quantum field theory by just proposing the commutation relations. This technique is an application of the idea of converting Poisson brackets in classical theory to commutators in quantum theory, as we described earlier. The commutators for quantum field theory are

$$\left[\phi^{a}(\boldsymbol{x},t),\phi^{b}(\boldsymbol{y},t)\right]=0 \tag{1}$$

$$\left[\pi_{a}(\boldsymbol{x},t),\pi_{b}(\boldsymbol{y},t)\right]=0 \tag{2}$$

$$\left[\phi^{a}(\boldsymbol{x},t),\pi_{b}(\boldsymbol{y},t)\right]=i\delta_{b}^{a}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}) \tag{3}$$

Note that these commutators are defined for equal times, but at possibly different spatial locations $\boldsymbol{x}$ and $\boldsymbol{y}$. Here

$$\pi_{a}=\frac{\partial\mathscr{L}}{\partial\dot{\phi}^{a}} \tag{4}$$

Coleman checks that these commutators are consistent by using them to calculate the equations of motion for the system with the Lagrangian and Hamiltonian:

$$\mathscr{L}=\frac{1}{2}\left(\partial_{\mu}\phi\partial^{\mu}\phi-\mu^{2}\phi^{2}\right) \tag{5}$$

$$\mathscr{H}=\frac{1}{2}\left(\pi^{2}+|\nabla\phi|^{2}+\mu^{2}\phi^{2}\right) \tag{6}$$

After going through the calculations, he obtains

$$\ddot{\phi}(\boldsymbol{x},t)=\nabla^{2}\phi(\boldsymbol{x},t)-\mu^{2}\phi(\boldsymbol{x},t) \tag{7}$$

which is the Klein-Gordon equation.

In section 4.5, Coleman uses the expansion of the scalar field in terms of creation and annihilation operators

$$\phi(\boldsymbol{x},t)=\int\frac{d^3\boldsymbol{p}}{\sqrt{(2\pi)^3}\sqrt{2\omega_{\boldsymbol{p}}}}\left(a_{\boldsymbol{p}}e^{-ip\cdot x}+a_{\boldsymbol{p}}^{\dagger}e^{ip\cdot x}\right) \tag{8}$$

to work out the total Hamiltonian, and then introduces normal ordering to eliminate the infinite constant in the Hamiltonian. There is nothing particularly new in this presentation. Normal ordering is denoted by the double colon:

$$:H: \tag{9}$$

and amounts to placing all creation operators to the left of all annihilation operators.

There is one caution that should be noted, however. Normal ordering should not be thought of as a process that can be applied to equations, since it can lead to contradictions. From the commutator

$$\left[a,a^{\dagger}\right]=1 \tag{10}$$

we can expand the equation to get

$$aa^{\dagger}=1+a^{\dagger}a \tag{11}$$

If we now 'apply' normal ordering to both sides, we get

$$:aa^{\dagger}:=a^{\dagger}a=1+a^{\dagger}a \tag{12}$$

or

$$1=0 \tag{13}$$

Normal ordering is something that is done to the result of a calculation and should not be used as an operation in an equation.