NOETHER'S THEOREM IN CLASSICAL MECHANICS

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Reference: Sidney Coleman, *Quantum Field Theory: Lectures of Sidney Coleman* (edited by Bryan Gin-ge Chen *et al.*), World Scientific, 2019. Section 5.1.

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Chapter 5 begins the study of symmetries and conservation laws, starting with Noether's theorem in classical mechanics. Although we've looked at Noether's theorem before (see here and here, for example), Coleman's treatment is by far the clearest I've seen so far, so it's worth summarizing. Note that everything in this post refers to classical particle mechanics, although the results can be extended to quantum theory.

We suppose that the set of generalized coordinates q^a are transformed in a way characterized by a real parameter λ, so that

The generalized coordinates are assumed to be functions of time t.

$$q^a(t) \to q^a(t;\lambda) \tag{1}$$

If we consider an infinitesimal transformation, then each coordinate will change according to a first order Taylor expansion:

$$q^a \to q^a + \left.\frac{\partial q^a}{\partial\lambda}\right|_{\lambda=0} d\lambda \tag{2}$$

Coleman gives the derivative a name:

$$Dq^a \equiv \left.\frac{\partial q^a}{\partial\lambda}\right|_{\lambda=0} \tag{3}$$

The change of the time derivatives of the coordinates is therefore

$$\dot{q}^a \to \dot{q}^a + \left.\frac{\partial \dot{q}^a}{\partial\lambda}\right|_{\lambda=0} d\lambda \tag{4}$$

$$= \dot{q}^a + D\dot{q}^a d\lambda \tag{5}$$

Since λ doesn't depend on time (for any given transformation, it's a constant) we have

$$D\dot{q}^a = \frac{d}{dt} Dq^a \tag{6}$$

Coleman now states that for a Lagrangian $L(q^a, \dot{q}^a, t)$ that depends on the coordinates, their time derivatives and possibly explicitly on the time, the change under the above transformation is given by DL. However, I think what he meant to say is that the amount by which the Lagrangian changes is $DL \times d\lambda$, with

$$DL = \frac{\partial L}{\partial q^a} Dq^a + \frac{\partial L}{\partial \dot{q}^a} D\dot{q}^a \tag{7}$$

$$= \frac{\partial L}{\partial q^a} Dq^a + p_a D\dot{q}^a \tag{8}$$

where

$$p_a \equiv \frac{\partial L}{\partial \dot{q}^a} \tag{9}$$

is the momentum conjugate to the coordinate q^a.

The quantity DL, as written, doesn't depend on $d\lambda$, so I think we need to explicitly include this when we want the change in the Lagrangian. That is

$$L \to L + DL\, d\lambda \tag{10}$$

We now call a transformation a *symmetry* if and only if

$$DL = \frac{dF}{dt} \tag{11}$$

for some function $F(q^a, \dot{q}^a, t)$. If this is true, then the action between two times t_1 and t_2 is unchanged, so the equations of motion should also be unchanged. Remember that the equations of motion were obtained from the principle of least action, so if we change the Lagrangian in a way that doesn't change the action, the equations of motion should remain the same. The new action is

$$\mathcal{S}' = \int_{t_1}^{t_2} (L + DL\, d\lambda)\, dt \tag{12}$$

$$= \int_{t_1}^{t_2} L\, dt + d\lambda \int_{t_1}^{t_2} DL\, dt \tag{13}$$

$$= \mathcal{S} + d\lambda \int_{t_1}^{t_2} \frac{dF}{dt} dt \tag{14}$$

$$= \mathcal{S} + d\lambda \left(F(q_2^a, \dot{q}_2^a, t_2) - F(q_1^a, \dot{q}_1^a, t_1)\right) \tag{15}$$

where q_1^a is the coordinate q^a at time t_1. One of the conditions imposed when calculating the least action is that the coordinates at the two extremes of time are fixed when the action is varied, so the the last term

$\left(F\left(q_2^a,\dot{q}_2^a,t_2\right)-F\left(q_1^a,\dot{q}_1^a,t_1\right)\right)$ is zero when varied. In other words, the conditions $\delta\mathcal{S}'=0$ and $\delta\mathcal{S}=0$ give the same equations of motion.

We can now derive Noether's theorem by showing that if 11 is satisfied, then there is a quantity Q that is conserved (that is, Q remains constant over time). We define

$$Q=p_aDq^a-F \tag{16}$$

and calculate its time derivative:

$$\frac{dQ}{dt}=\dot{p}_aDq^a+p_a\frac{dDq^a}{dt}-\frac{dF}{dt} \tag{17}$$

We now use the Euler-Lagrange equation for classical particle mechanics:

$$\frac{\partial L}{\partial q^a}=\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}^a}\right) \tag{18}$$

Using 9, this is equivalent to

$$\dot{p}_a=\frac{\partial L}{\partial q^a} \tag{19}$$

so 17 becomes

$$\frac{dQ}{dt}=\frac{\partial L}{\partial q^a}Dq^a+p_a\frac{dDq^a}{dt}-\frac{dF}{dt} \tag{20}$$

From 8, we see that the first two terms on the RHS are DL so from that and 11 we have

$$\frac{dQ}{dt}=DL-DL=0 \tag{21}$$

so that the quantity Q is conserved.

Coleman gives 3 examples of applying Noether's theorem. It's worth going over the first example in a bit more detail.

He uses the Lagrangian

$$L=\frac{1}{2}\sum_r m_r\dot{\boldsymbol{x}}^r\cdot\dot{\boldsymbol{x}}^r+\sum_{r>s}V^{(r,s)}\left(\left|\boldsymbol{x}^r-\boldsymbol{x}^s\right|\right) \tag{22}$$

This describes a system with the usual classical kinetic energy in the first term and a potential energy $V^{(r,s)}$ that depends only on the distances between the particles, and not on their absolute positions. If we consider the transformation

$$\boldsymbol{x}^r\to\boldsymbol{x}^r+\lambda\boldsymbol{e} \tag{23}$$

where $\boldsymbol{e}$ is a fixed unit vector in some given direction, then we have translated the entire system by a fixed amount in that direction. It's fairly obvious from the Lagrangian 22 that it won't change under this transformation, since $\dot{\boldsymbol{x}}^r$ (the derivative of $\boldsymbol{x}^r$) won't change, and, since all particles are translated by exactly the same amount, the distances between them won't change either.

However, just for peace of mind, it's worth doing the explicit calculation of DL in 8. To simplify things, we'll consider the case where we have only 2 particles. The generalization to the multi-particle 3-d case is straightforward, if a bit messy. In this case, the Lagrangian is

$$L=\frac{1}{2}\sum_{r=1}^{2}\left[m_r\left(\dot{x}^r\right)^2+\left(\dot{y}^r\right)^2+\left(\dot{z}^r\right)^2\right]+V\left(\sqrt{\left(x^1-x^2\right)^2+\left(y^1-y^2\right)^2+\left(z^1-z^2\right)^2}\right) \tag{24}$$

From 23 we have

$$\begin{aligned} Dx^r &= \left.\frac{\partial x^r}{\partial\lambda}\right|_{\lambda=0} && (25)\\ &= e_1 && (26)\end{aligned}$$

and similarly for the other coordinates. Here, the superscript r indicates particle r, and the subscript 1 indicates the x component the vector $\boldsymbol{e}$.

Since

$$\dot{\boldsymbol{x}}^r \to \dot{\boldsymbol{x}}^r \tag{27}$$

we have

$$D\dot{x}^1 = \left.\frac{\partial \dot{x}^1}{\partial\lambda}\right|_{\lambda=0} = 0 \tag{28}$$

We also have, after defining

$$r \equiv \sqrt{\left(x^1-x^2\right)^2+\left(y^1-y^2\right)^2+\left(z^1-z^2\right)^2} \tag{29}$$

the following, using the chain rule:

$$\begin{aligned} \frac{\partial L}{\partial x^1} &= \frac{\partial V}{\partial r}\frac{\left(x^1-x^2\right)}{r} && (30)\\ \frac{\partial L}{\partial x^2} &= -\frac{\partial V}{\partial r}\frac{\left(x^1-x^2\right)}{r} && (31)\end{aligned}$$

Also

$$\frac{\partial L}{\partial \dot{x}^1} = m_1 \dot{x}^1 \tag{32}$$

and likewise for the other coordinates.

Therefore

$$DL = \frac{\partial L}{\partial q^a} Dq^a + \frac{\partial L}{\partial \dot{q}^a} D\dot{q}^a \tag{33}$$

$$= \frac{\partial V}{\partial r} \frac{\left(x^1 - x^2\right)}{r} e_1 - \frac{\partial V}{\partial r} \frac{\left(x^1 - x^2\right)}{r} e_1 + \ldots + 0 \tag{34}$$

where the $\ldots$ in the second line are the terms involving V for the other coordinates, and the $+0$ is for the term $\frac{\partial L}{\partial \dot{q}^a} D\dot{q}^a$, since $D\dot{q}^a = 0$.

We can see that terms in DL all cancel in pairs, giving the result $DL = 0$ (which is what we'd expect from the physics of the situation as described above). Since $DL = 0$, then taking $F = 0$ satisfies 11.

From 16 and 32 we can now find the conserved quantity:

$$Q = p_a Dq^a \tag{35}$$

$$= \left(\frac{\partial L}{\partial \dot{x}^1} + \frac{\partial L}{\partial \dot{x}^2}\right) e_1 + \left(\frac{\partial L}{\partial \dot{y}^1} + \frac{\partial L}{\partial \dot{y}^2}\right) e_2 + \left(\frac{\partial L}{\partial \dot{z}^1} + \frac{\partial L}{\partial \dot{z}^2}\right) e_3 \tag{36}$$

$$= \left(m_1 \dot{x}^1 + m_2 \dot{x}^2\right) e_1 + \left(m_1 \dot{y}^1 + m_2 \dot{y}^2\right) e_2 + \left(m_1 \dot{z}^1 + m_2 \dot{z}^2\right) e_3 \tag{37}$$

$$= m_1 \dot{\boldsymbol{x}}^1 \cdot \boldsymbol{e} + m_2 \dot{\boldsymbol{x}}^2 \cdot \boldsymbol{e} \tag{38}$$

Since the vector $\boldsymbol{e}$ is arbitrary, we get an infinite number of conserved quantities, one for each direction of $\boldsymbol{e}$. However, in 3-d space there are only 3 independent directions (one for each coordinate axis), so we actually get only 3 independent conserved quantities. These quantities can be combined into the single vector

$$\boldsymbol{p} = \sum_r m_r \dot{\boldsymbol{x}}^r \tag{39}$$

which is just the total classical momentum. In other words, for a Lagrangian that is symmetric under translation, the corresponding system has a conserved total momentum.

Coleman's derivation is, of course, much shorter, but it relies on physical intuition rather than explicitly working out the components of DL, so I think it's worth going through the gory details just to see how the system works.

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