NOETHER'S THEOREM IN CLASSICAL FIELD THEORY

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Reference: Sidney Coleman, *Quantum Field Theory: Lectures of Sidney Coleman* (edited by Bryan Gin-ge Chen *et al.*), World Scientific, 2019. Sections 5.3-5.4.

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To apply Noether's theorem to classical field theory, we can extend the ideas developed in classical particle mechanics. The dynamical variables have migrated from the coordinates of particles to fields that are defined as functions of space and time. We therefore define a transformation by introducing a parameter λ which governs the size of the transformation, so that

$$\phi^{a}(x)\rightarrow\phi^{a}(x,\lambda) \tag{1}$$

By analogy with the particle case, the change in the field due to an infinitesimal change $d\lambda$ is

$$\phi^{a}(x)\rightarrow\phi^{a}(x)+\left.\frac{\partial\phi^{a}}{\partial\lambda}\right|_{\lambda=0}d\lambda \tag{2}$$

so we can define the quantity

$$D\phi^{a}\equiv\left.\frac{\partial\phi^{a}}{\partial\lambda}\right|_{\lambda=0} \tag{3}$$

For a Lagrangian density $\mathscr{L}(\phi^{a},\partial_{\mu}\phi^{a},x)$ we can define the quantity

$$D\mathscr{L}=\frac{\partial L}{\partial\phi^{a}}D\phi^{a}+\frac{\partial L}{\partial(\partial_{\mu}\phi^{a})}D\partial_{\mu}\phi^{a} \tag{4}$$

$$=\frac{\partial L}{\partial\phi^{a}}D\phi^{a}+\pi_{a}^{\mu}\partial_{\mu}(D\phi^{a}) \tag{5}$$

with

$$\pi_{a}^{\mu}=\frac{\partial L}{\partial(\partial_{\mu}\phi^{a})} \tag{6}$$

as before.

Under the transformation $d\lambda$, the Lagrangian therefore transforms as

$$\mathscr{L} \to \mathscr{L} + D\mathscr{L}\, d\lambda \tag{7}$$

In the particle case, we specified that an infinitesimal transformation was a symmetry if $DL = dF/dt$. In the field theory case, the fields are functions of all four coordinates of spacetime rather than just the time, so we generalize the definition of a symmetry in field theory to be a transformation such that

$$D\mathscr{L} = \partial_\mu F^\mu \tag{8}$$

for some four-component object

$$F^\mu = F^\mu \left(\phi^a, \partial_\mu \phi^a, x\right) \tag{9}$$

In the particle case, we defined a quantity Q as

$$Q = p_a D q^a - F \tag{10}$$

In the field case, we define

$$J^\mu \equiv \pi_a^\mu D\phi^a - F^\mu \tag{11}$$

Coleman shows in his equations 5.27 to 5.29 that this quantity has a zero divergence:

$$\partial_\mu J^\mu = 0 \tag{12}$$

Using Coleman's convention for ∂_μ (his eqn 1.19) as

$$\partial_\mu = \left(\frac{\partial}{\partial t}, \nabla\right) \tag{13}$$

12 can be expanded to

$$\partial_0 J^0 + \nabla \cdot \boldsymbol{J} = 0 \tag{14}$$

where $\boldsymbol{J}$ is the 3-vector spatial part of J^μ. Integrating this over some fixed volume V we get from Gauss's theorem the condition

$$\partial_0 \int_V d^3\boldsymbol{x}\; J^0 = -\int_V d^3\boldsymbol{x}\; \nabla \cdot \boldsymbol{J} \tag{15}$$

$$= -\int_S d^2S\; \hat{\boldsymbol{n}} \cdot \boldsymbol{J} \tag{16}$$

where S is the surface bounding V. In other words, the flow of $\boldsymbol{J}$ across the boundary (note the negative sign) is balanced by the rate of change of J^0 inside V. We can therefore think of J^0 as some conserved 'charge'-like quantity and $\boldsymbol{J}$ as the current of this quantity. If we assume that the current falls to zero at infinity, then the current across the surface at infinity is zero, so

$$\partial_0 \int d^3\boldsymbol{x}\ J^0 = 0 \tag{17}$$

That is, the total quantity $\int_V d^3\boldsymbol{x}\ J^0$ is conserved.

In section 5.4, Coleman points out that the quantity J^μ is not unique for any given transformation. This is because the quantity F^μ was defined in 8 only by its divergence, so we can add any function to F^μ provided what we add has a zero divergence. In fact, if we add the divergence of some quantity $A^{\mu\nu}$ where $A^{\mu\nu} = -A^{\nu\mu}$ (that is, it's antisymmetric), then we have

$$F^\mu \to F^\mu + \partial_\nu A^{\mu\nu} \tag{18}$$

and

$$\partial_\mu F^\mu \to \partial_\mu F^\mu + \partial_\mu \partial_\nu A^{\mu\nu} \tag{19}$$

Because of the antisymmetry of $A^{\mu\nu}$, and the fact that $\partial_\mu\partial_\nu = \partial_\nu\partial_\mu$, we have

$$\partial_\mu\partial_\nu A^{\mu\nu} = \partial_\nu\partial_\mu A^{\mu\nu} \tag{20}$$

$$= -\partial_\nu\partial_\mu A^{\nu\mu} \tag{21}$$

$$= -\partial_\mu\partial_\nu A^{\mu\nu} \tag{22}$$

where in the last line, we've just swapped the dummy indices $\mu \leftrightarrow \nu$, since they are both summed and it doesn't matter what we call them. Thus $\partial_\mu\partial_\nu A^{\mu\nu}$ is equal to its own negative, so it must be equal to zero, which means that $\partial_\mu F^\mu$ is unchanged by the addition in 18.

Making this change also changes J^μ as we see from 11

$$J^\mu \to J^\mu - \partial_\nu A^{\mu\nu} \tag{23}$$

However, this doesn't change the total amount of 'charge' as specified by the integral of J^0 since we now have

$$\int d^3\boldsymbol{x}\ J^0 = \int d^3\boldsymbol{x}\ \left(J^0 - \partial_\nu A^{0\nu}\right) \tag{24}$$

$$= \int d^3\boldsymbol{x}\ J^0 - \int d^3\boldsymbol{x}\ \partial_\nu A^{0\nu} \tag{25}$$

$$= \int d^3\boldsymbol{x}\ J^0 - \int d^3\boldsymbol{x}\ \partial_i A^{0i} \tag{26}$$

I think the + sign in Coleman's eqn 5.37 should be a $-$ because of eqn 13, although it doesn't change the argument.

We can replace the ν index in the second line by i (over space indexes only) since the antisymmetry of $A^{\mu\nu}$ means that $A^{00} = 0$. The second integral in the last line is now the volume integral of a divergence, so by Gauss's theorem, we convert this to a surface integral at infinity and set it to zero. Thus $\int d^3\boldsymbol{x}\ J^0$ remains unchanged.

Thus the overall result is that in field theory we get a *local* conservation law, given by 12. Recall that J^μ is a set of densities, so is defined locally as functions of spacetime. The local amount of J^0 is balanced by a current given by $\boldsymbol{J}$ which causes J^0 'stuff' to flow in or out of a given volume. By integrating over all space, we also get a global conservation law, showing that the total amount of J^0 (whatever that substance may be) remains constant.

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