CURRENTS FROM SPACETIME TRANSLATIONS AND THE ENERGY-MOMENTUM TENSOR

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Reference: Sidney Coleman, *Quantum Field Theory: Lectures of Sidney Coleman* (edited by Bryan Gin-ge Chen *et al.*), World Scientific, 2019. Section 5.5.

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In Section 5.5, Coleman applies Noether's theorem in classical field theory to the case of spacetime translation. The transformation is applied to the fields, so we have

$$\phi^{a}(x)\rightarrow\phi^{a}(x+\lambda e) \tag{1}$$

where λ is the parameter giving the size of the translation and e is a constant 4-vector in spacetime giving the direction of the translation. We wish to find the current given by

$$J^{\mu}\equiv\pi_{a}^{\mu}D\phi^{a}-F^{\mu} \tag{2}$$

with the quantities defined as

$$D\phi^{a}\equiv\left.\frac{\partial\phi^{a}}{\partial\lambda}\right|_{\lambda=0} \tag{3}$$

$$D\mathscr{L}=\partial_{\mu}F^{\mu} \tag{4}$$

$$\pi_{a}^{\mu}=\frac{\partial\mathscr{L}}{\partial\left(\partial_{\mu}\phi^{a}\right)} \tag{5}$$

From 1 we have

$$D\phi^{a}=e_{\rho}\partial^{\rho}\phi^{a}(x) \tag{6}$$

Since we're considering infinitesimal translations, all the derivative terms $D\ldots$ depend linearly on the components of the vector e, so we will expect the currents J^{μ} to be some linear combination of e_{ρ}. The most general such expression is

$$J^{\mu}=e_{\rho}T^{\rho\mu} \tag{7}$$

where $T^{\rho\mu}$ is, at this stage, nothing more than a general 4×4 matrix.

We saw earlier that

$$\partial_\mu J^\mu = 0 \tag{8}$$

so from 7 this gives us (since the components e_ρ are arbitrary)

$$\partial_\mu T^{\rho\mu} = 0 \tag{9}$$

The quantity $T^{\rho\mu}$ is known as the *canonical energy-momentum tensor*.

At this stage, it's useful to compare this result with that for classical particle theory, where the system was translated in space only by $\boldsymbol{x} \rightarrow \boldsymbol{x} + \lambda \boldsymbol{e}$. In that case, Noether's theorem resulted in a global conservation law for the quantity

$$Q = \boldsymbol{e} \cdot \sum_r m_r \dot{\boldsymbol{x}}^r \tag{10}$$

$$= \boldsymbol{e} \cdot \boldsymbol{p} \tag{11}$$

where $\boldsymbol{p}$ is the total momentum. Remember that in particle mechanics, the conservation laws are global, applying to the system as a whole, but in field theory, the currents J^μ are *densities*, and we get both a local and a global conservation law.

Returning to field theory, we observe that, so far, the only condition we have on $T^{\rho\mu}$ is 9, and therefore $T^{\rho\mu}$ is not unique. This is because we can add the divergence of some antisymmetric object $A^{\rho\mu\lambda}$, where the antisymmetry applies to the last two indices:

$$A^{\rho\mu\lambda} = -A^{\rho\lambda\mu} \tag{12}$$

We can therefore use a modified energy-momentum tensor of form

$$\theta^{\rho\mu} = T^{\rho\mu} + \partial_\lambda A^{\rho\mu\lambda} \tag{13}$$

because the divergence with respect to μ of the last term is zero:

$$\partial_\mu \partial_\lambda A^{\rho\mu\lambda} = 0 \tag{14}$$

The tensor $T^{\rho\mu}$ contains 16 quantities. We can get some insight into what these components mean physically by applying the integral form of the conservation law 8. From 7, suppose we take

$$e = (1, 0, 0, 0) \tag{15}$$

That is, e represents a translation in time only. Then we have

$$J^\mu = T^{0\mu} \tag{16}$$

and from 8 we have

$$\partial_0 T^{00} + \partial_i T^{0i} = 0 \tag{17}$$

Integrating this over a volume V with surface S and using Gauss's theorem, we get

$$\frac{d}{dt}\int_V d^3\boldsymbol{x}\, T^{00} = -\int_V d^3\boldsymbol{x}\, \partial_i T^{0i} \tag{18}$$

$$= -\int_S d^2\boldsymbol{x}\, n_i T^{0i} \tag{19}$$

where $\boldsymbol{n}$ is a unit normal to the surface.

In this case, the 'charge'-like quantity is the integral of T^{00} and the 'current'-like quantity is T^{0i}. We see that the change in the integral of T^{00} is given by the integral of the flow of this 'stuff' across the surface. Coleman shows that T^{00} is in fact the energy density, so the quantities T^{0i} are the current of energy in each of the three coordinate directions.

We have a similar interpretation for the other components, although in this case, it is momentum density and momentum flow. For example, if

$$e = (0,1,0,0) \tag{20}$$

so that the system is translated along the x^1 axis only, we get

$$\frac{d}{dt}\int_V d^3\boldsymbol{x}\, T^{10} = -\int_V d^3\boldsymbol{x}\, \partial_i T^{1i} \tag{21}$$

$$= -\int_S d^2\boldsymbol{x}\, n_i T^{1i} \tag{22}$$

In this case, the conserved 'stuff' is the total momentum in the x^1 direction, given by $\int_V d^3\boldsymbol{x}\, T^{10}$, and the quantities T^{1i} represent the flow, or current, of x^1 momentum in each of the three coordinate directions.

To complete the derivation, Coleman derives an explicit form for J^μ using 2 (details in his eqns 5.46 to 5.50) with the result that

$$J^\mu = \pi_a^\mu e_\rho \partial^\rho \phi^a - g^{\mu\rho} e_\rho \mathscr{L} \tag{23}$$

where $g^{\mu\rho}$ is the metric tensor. Comparing this with 7 and using the fact that e_ρ is arbitrary so we can factor it out, we get an explicit form for the energy-momentum tensor:

$$T^{\rho\mu} = \pi_a^\mu \partial^\rho \phi^a - g^{\mu\rho} \mathscr{L} \tag{24}$$

Although all this was derived for *classical* field theory, it turns out to be valid for quantum field theory as well. Coleman goes through the derivation for the scalar quantum field

$$\phi(\boldsymbol{x},t)=\int\frac{d^{3}\boldsymbol{p}}{\sqrt{(2\pi)^{3}}\sqrt{2\omega_{\boldsymbol{p}}}}\left(a_{\boldsymbol{p}}e^{-ip\cdot x}+a_{\boldsymbol{p}}^{\dagger}e^{ip\cdot x}\right) \tag{25}$$

He shows that, with a bit of mathematical licence (juggling an infinite delta function) or by using normal ordering, we can derive the conserved momentum for the quantum field, giving

$$\boldsymbol{P}=\int d^{3}\boldsymbol{p}\,\left(a_{\boldsymbol{p}}^{\dagger}a_{\boldsymbol{p}}\right)\boldsymbol{p} \tag{26}$$

PINGBACKS

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