CONTINUOUS INTERNAL SYMMETRIES

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Reference: Sidney Coleman, *Quantum Field Theory: Lectures of Sidney Coleman* (edited by Bryan Gin-ge Chen *et al.*), World Scientific, 2019. Section 6.1.

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In section 6.1, Coleman introduces *internal symmetries*, which are symmetries of a Lagrangian that aren't due to any external transformation, such as spacetime translation or a Lorentz transformation. He illustrates the theory with the Lagrangian

$$\mathscr{L}=\frac{1}{2}\left(\partial^{\mu}\phi^{a}\partial_{\mu}\phi^{a}-\mu^{2}\phi^{a}\phi^{a}\right) \tag{1}$$

where $a=1,2$ so there are two distinct fields. He observes that if introduce the transformation given by

$$\phi^{1}\rightarrow\phi^{1}\cos\lambda+\phi^{2}\sin\lambda \tag{2}$$

$$\phi^{2}\rightarrow\phi^{2}\cos\lambda-\phi^{1}\sin\lambda \tag{3}$$

the Lagrangian 1 is unchanged, as we can verify by just substituting the transformation into $\mathscr{L}$ and multiplying them out. This symmetry has nothing to do with a physical external transformation of the system, so it's known as an internal symmetry.

We can now extract the conserved current J^{μ} corresponding to this symmetry by applying the general procedure we worked out earlier. That is, we want to find

$$J^{\mu}\equiv\pi_{a}^{\mu}D\phi^{a}-F^{\mu} \tag{4}$$

where

$$D\phi^{a}\equiv\left.\frac{\partial\phi^{a}}{\partial\lambda}\right|_{\lambda=0} \tag{5}$$

$$\pi_{a}^{\mu}=\frac{\partial L}{\partial\left(\partial_{\mu}\phi^{a}\right)} \tag{6}$$

$$D\mathscr{L}=\partial_{\mu}F^{\mu} \tag{7}$$

Coleman works these out in his eqns 6.4 to 6.7 with the result that

$$J^{\mu}=\left(\partial^{\mu}\phi^{1}\right)\phi^{2}-\left(\partial^{\mu}\phi^{2}\right)\phi^{1} \tag{8}$$

Using the usual expansion for a scalar field

$$\phi(x)=\int\frac{d^{3}\boldsymbol{p}}{\sqrt{(2\pi)^{3}}\sqrt{2\omega_{\boldsymbol{p}}}}\left(a_{\boldsymbol{p}}e^{-ip\cdot x}+a_{\boldsymbol{p}}^{\dagger}e^{ip\cdot x}\right) \tag{9}$$

we now expand this definition to allow for 2 distinct fields, so we have

$$\phi^{a}(x)=\int\frac{d^{3}\boldsymbol{p}}{\sqrt{(2\pi)^{3}}\sqrt{2\omega_{\boldsymbol{p}}}}\left(a_{\boldsymbol{p}}^{(a)}e^{-ip\cdot x}+a_{\boldsymbol{p}}^{(a)\dagger}e^{ip\cdot x}\right) \tag{10}$$

The conserved charge Q is given by

$$Q=\int d^{3}\boldsymbol{x}\ J^{0} \tag{11}$$

so we can plug the fields 10 into 8 and do the integrals. This follows the same procedure as Coleman used in working out similar quantities in Chapters 4 - see eqns 4.57 to 4.62, and 5.53 to 5.58 for the details. Basically it involves producing delta functions to get rid of some of the integrals, and observing that any remaining terms with a time dependence cancel out, so the net value of Q is independent of time (and of course space, since we've integrated over all space). The value of Q comes out to eqn 6.11:

$$Q=i\int d^{3}\boldsymbol{p}\ \left[a_{\boldsymbol{p}}^{(1)\dagger}a_{\boldsymbol{p}}^{(2)}-a_{\boldsymbol{p}}^{(2)\dagger}a_{\boldsymbol{p}}^{(1)}\right] \tag{12}$$

Coleman observes that Q in this form commutes with the energy H which in this case is the sum of two terms, one for each particle type:

$$H=\int d^{3}\boldsymbol{p}\ \left[a_{\boldsymbol{p}}^{(1)\dagger}a_{\boldsymbol{p}}^{(1)}+a_{\boldsymbol{p}}^{(2)\dagger}a_{\boldsymbol{p}}^{(2)}\right]\omega_{\boldsymbol{p}} \tag{13}$$

We can demonstrate the commutator by using the commutators for the $a_{\boldsymbol{p}}$ operators:

$$\left[a_{\boldsymbol{p}}^{(a)},a_{\boldsymbol{p}'}^{(b)}\right]=\left[a_{\boldsymbol{p}}^{(a)\dagger},a_{\boldsymbol{p}'}^{(b)\dagger}\right]=0 \tag{14}$$

$$\left[a_{\boldsymbol{p}}^{(a)},a_{\boldsymbol{p}'}^{(b)\dagger}\right]=\delta^{ab}\delta^{(3)}\left(\boldsymbol{p}-\boldsymbol{p}'\right) \tag{15}$$

For example, one of the commutators we need in working out $[Q,H]$ is

$$\left[a_{\boldsymbol{p}}^{(1)\dagger}a_{\boldsymbol{p}}^{(2)},a_{\boldsymbol{p}'}^{(1)\dagger}a_{\boldsymbol{p}'}^{(1)}\right]=\left[a_{\boldsymbol{p}}^{(1)\dagger},a_{\boldsymbol{p}'}^{(1)\dagger}a_{\boldsymbol{p}'}^{(1)}\right]a_{\boldsymbol{p}}^{(2)} \tag{16}$$

$$=\left(a_{\boldsymbol{p}}^{(1)\dagger}a_{\boldsymbol{p}'}^{(1)\dagger}a_{\boldsymbol{p}'}^{(1)}-a_{\boldsymbol{p}'}^{(1)\dagger}a_{\boldsymbol{p}'}^{(1)}a_{\boldsymbol{p}}^{(1)\dagger}\right)a_{\boldsymbol{p}}^{(2)} \tag{17}$$

$$=\left(a_{\boldsymbol{p}}^{(1)\dagger}a_{\boldsymbol{p}'}^{(1)\dagger}a_{\boldsymbol{p}'}^{(1)}-a_{\boldsymbol{p}}^{(1)\dagger}a_{\boldsymbol{p}'}^{(1)\dagger}a_{\boldsymbol{p}'}^{(1)}-\delta^{(3)}\left(\boldsymbol{p}-\boldsymbol{p}'\right)a_{\boldsymbol{p}'}^{(1)\dagger}\right)a_{\boldsymbol{p}}^{(2)} \tag{18}$$

$$=-\delta^{(3)}\left(\boldsymbol{p}-\boldsymbol{p}'\right)a_{\boldsymbol{p}'}^{(1)\dagger}a_{\boldsymbol{p}}^{(2)} \tag{19}$$

By the same calculation, we can work out

$$\left[a_{\boldsymbol{p}}^{(1)\dagger}a_{\boldsymbol{p}}^{(2)},a_{\boldsymbol{p}'}^{(2)\dagger}a_{\boldsymbol{p}'}^{(2)}\right]=\delta^{(3)}\left(\boldsymbol{p}-\boldsymbol{p}'\right)a_{\boldsymbol{p}'}^{(1)\dagger}a_{\boldsymbol{p}}^{(2)} \tag{20}$$

so the sum of 19 and 20 gives zero. The commutator of the last term in 12 with H also gives zero, so

$$[Q,H]=0 \tag{21}$$

Because Q has annihilation operators on the right in both terms in 12, it annihilates the vacuum

$$Q\left|0\right\rangle=0 \tag{22}$$

By applying the same commutation rules as before, we can also work out the commutators of Q with the $a_{\boldsymbol{p}}$ and $a_{\boldsymbol{p}}^{\dagger}$ operators, and we get Coleman's eqn 6.13:

$$\left[Q,a_{\boldsymbol{p}}^{(a)}\right]=-i\epsilon^{ab}a_{\boldsymbol{p}}^{(b)} \tag{23}$$

$$\left[Q,a_{\boldsymbol{p}}^{(a)\dagger}\right]=-i\epsilon^{ab}a_{\boldsymbol{p}}^{(b)\dagger} \tag{24}$$

where $\epsilon^{12}=-\epsilon^{21}=1$ and $\epsilon^{11}=\epsilon^{22}=0$.

The states containing a certain number of particles of type 1 or type 2 are not eigenstates of Q, as we can see by direct calculation. For example, if we start with a state containing one type 1 particle $\left|1\right\rangle$, we have

$$Q\left|1_{\boldsymbol{p}'}\right\rangle = i\int d^3\boldsymbol{p}\ \left[a_{\boldsymbol{p}}^{(1)\dagger}a_{\boldsymbol{p}}^{(2)}\left|1_{\boldsymbol{p}'}\right\rangle - a_{\boldsymbol{p}}^{(2)\dagger}a_{\boldsymbol{p}}^{(1)}\left|1_{\boldsymbol{p}'}\right\rangle\right] \tag{25}$$

$$= i\int d^3\boldsymbol{p}\ \left[0 - a_{\boldsymbol{p}}^{(2)\dagger}a_{\boldsymbol{p}}^{(1)}a_{\boldsymbol{p}}^{(1)\dagger}\left|0\right\rangle\right] \tag{26}$$

$$= -i\int d^3\boldsymbol{p}\ -a_{\boldsymbol{p}}^{(2)\dagger}\delta^{(3)}\left(\boldsymbol{p}-\boldsymbol{p}'\right)\left|0\right\rangle \tag{27}$$

$$= -ia_{\boldsymbol{p}'}^{(2)\dagger}\left|0\right\rangle \tag{28}$$

$$= -i\left|2_{\boldsymbol{p}'}\right\rangle \tag{29}$$

The second line follows because $a_{\boldsymbol{p}}^{(2)}$ acting on any state with no type 2 particles gives zero, and the third line uses the commutator 15. Thus Q acting on a type 1 particle gives $-i$ times a state with a type 2 particle, so $\left|1_{\boldsymbol{p}}\right\rangle$ is not an eigenstate of Q.

Coleman then shows that if we use a linear combination of the states we get a better form for Q. These new states are

$$b_{\boldsymbol{p}} = \frac{1}{\sqrt{2}}\left(a_{\boldsymbol{p}}^{(1)} + ia_{\boldsymbol{p}}^{(2)}\right) \tag{30}$$

$$b_{\boldsymbol{p}}^{\dagger} = \frac{1}{\sqrt{2}}\left(a_{\boldsymbol{p}}^{(1)\dagger} - ia_{\boldsymbol{p}}^{(2)\dagger}\right) \tag{31}$$

$$c_{\boldsymbol{p}} = \frac{1}{\sqrt{2}}\left(a_{\boldsymbol{p}}^{(1)} - ia_{\boldsymbol{p}}^{(2)}\right) \tag{32}$$

$$c_{\boldsymbol{p}}^{\dagger} = \frac{1}{\sqrt{2}}\left(a_{\boldsymbol{p}}^{(1)\dagger} + ia_{\boldsymbol{p}}^{(2)\dagger}\right) \tag{33}$$

We can work out the commutators by grinding through the calculations using 15 and we find

$$\left[b_{\boldsymbol{p}}, b_{\boldsymbol{p}'}^{\dagger}\right] = \left[c_{\boldsymbol{p}}, c_{\boldsymbol{p}'}^{\dagger}\right] = \delta^{(3)}\left(\boldsymbol{p}-\boldsymbol{p}'\right) \tag{34}$$

with all other commutators being zero.

By substitution we find that

$$\begin{aligned} b_{\boldsymbol{p}}^{\dagger}b_{\boldsymbol{p}} - c_{\boldsymbol{p}}^{\dagger}c_{\boldsymbol{p}} &= \frac{1}{2}\left(a_{\boldsymbol{p}}^{(1)\dagger} - ia_{\boldsymbol{p}}^{(2)\dagger}\right)\left(a_{\boldsymbol{p}}^{(1)} + ia_{\boldsymbol{p}}^{(2)}\right) \\ &\quad - \frac{1}{2}\left(a_{\boldsymbol{p}}^{(1)\dagger} + ia_{\boldsymbol{p}}^{(2)\dagger}\right)\left(a_{\boldsymbol{p}}^{(1)} - ia_{\boldsymbol{p}}^{(2)}\right) \end{aligned} \tag{35}$$

$$= i\left(a_{\boldsymbol{p}}^{(1)\dagger}a_{\boldsymbol{p}}^{(2)} - a_{\boldsymbol{p}}^{(2)\dagger}a_{\boldsymbol{p}}^{(1)}\right) \tag{36}$$

Thus we can rewrite Q from 12 as

$$Q = \int d^3\boldsymbol{p}\ \left[b^\dagger_{\boldsymbol{p}} b_{\boldsymbol{p}} - c^\dagger_{\boldsymbol{p}} c_{\boldsymbol{p}}\right] \tag{37}$$

$$= N_b - N_c \tag{38}$$

where N_b and N_c are the number operators for particles of type b and c.

Again, by direct calculation, we can show that states with particles of type b and c are now eigenstates of Q. For example, if we start with a state with a single b particle, then

$$Q\left|b_{\boldsymbol{p}'}\right\rangle = \int d^3\boldsymbol{p}\ \left[b^\dagger_{\boldsymbol{p}} b_{\boldsymbol{p}}\left|b_{\boldsymbol{p}'}\right\rangle - c^\dagger_{\boldsymbol{p}} c_{\boldsymbol{p}}\left|b_{\boldsymbol{p}'}\right\rangle\right] \tag{39}$$

$$= \int d^3\boldsymbol{p}\ \left[b^\dagger_{\boldsymbol{p}} b_{\boldsymbol{p}} b^\dagger_{\boldsymbol{p}'}\left|0\right\rangle - 0\right] \tag{40}$$

$$= \int d^3\boldsymbol{p}\ b^\dagger_{\boldsymbol{p}}\delta^{(3)}\left(\boldsymbol{p}-\boldsymbol{p}'\right)\left|0\right\rangle \tag{41}$$

$$= b^\dagger_{\boldsymbol{p}'}\left|0\right\rangle \tag{42}$$

$$= \left|b_{\boldsymbol{p}'}\right\rangle \tag{43}$$

We could also arrive at this same conclusion simply by using 38, since Q acting on a state containing a number of b and c particles gives the number of b particles minus the number of c particles times the original state.

By using the commutators 34 we can also work out the commutators of Q with the b and c operators, which Coleman gives in eqns 6.20 and 6.21:

$$[Q, b_{\boldsymbol{p}}] = -b_{\boldsymbol{p}} \tag{44}$$

$$\left[Q, b^\dagger_{\boldsymbol{p}}\right] = b^\dagger_{\boldsymbol{p}} \tag{45}$$

$$[Q, c_{\boldsymbol{p}}] = c_{\boldsymbol{p}} \tag{46}$$

$$\left[Q, c^\dagger_{\boldsymbol{p}}\right] = -c^\dagger_{\boldsymbol{p}} \tag{47}$$

One thing in this section does mystify me a bit, however. Although it's straightforward enough to work out the commutators and show that Q is diagonal with respect to states of b and c particles, Coleman seems to imply that the commutators 44 also show that Q is diagonal. I'm not aware of any connection between a matrix being diagonal and these commutators, so if anyone can shed some light on this, please do leave a comment.

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