COMPLEX FIELDS

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Reference: Sidney Coleman, *Quantum Field Theory: Lectures of Sidney Coleman* (edited by Bryan Gin-ge Chen *et al.*), World Scientific, 2019. Section 6.1.

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We've looked at a system of two fields with the Lagrangian

$$\mathscr{L}=\frac{1}{2}\left(\partial^{\mu}\phi^{a}\partial_{\mu}\phi^{a}-\mu^{2}\phi^{a}\phi^{a}\right) \tag{1}$$

where $a=1,2$. We saw that this Lagrangian is invariant under the transformation

$$\phi^{1}\rightarrow\phi^{1}\cos\lambda+\phi^{2}\sin\lambda \tag{2}$$

$$\phi^{2}\rightarrow\phi^{2}\cos\lambda-\phi^{1}\sin\lambda \tag{3}$$

for some parameter λ. We also saw that the conserved 'charge' Q is most conveniently expressed in terms of linear combinations of the creation and annihilation operators, in the form

$$Q=\int d^{3}\boldsymbol{p}\left[b_{\boldsymbol{p}}^{\dagger}b_{\boldsymbol{p}}-c_{\boldsymbol{p}}^{\dagger}c_{\boldsymbol{p}}\right] \tag{4}$$

$$=N_{b}-N_{c} \tag{5}$$

where

$$b_{\boldsymbol{p}}=\frac{1}{\sqrt{2}}\left(a_{\boldsymbol{p}}^{(1)}+ia_{\boldsymbol{p}}^{(2)}\right) \tag{6}$$

$$b_{\boldsymbol{p}}^{\dagger}=\frac{1}{\sqrt{2}}\left(a_{\boldsymbol{p}}^{(1)\dagger}-ia_{\boldsymbol{p}}^{(2)\dagger}\right) \tag{7}$$

$$c_{\boldsymbol{p}}=\frac{1}{\sqrt{2}}\left(a_{\boldsymbol{p}}^{(1)}-ia_{\boldsymbol{p}}^{(2)}\right) \tag{8}$$

$$c_{\boldsymbol{p}}^{\dagger}=\frac{1}{\sqrt{2}}\left(a_{\boldsymbol{p}}^{(1)\dagger}+ia_{\boldsymbol{p}}^{(2)\dagger}\right) \tag{9}$$

At this point, we might ask if it would make sense to define fields that are linear combinations of the fields for particles of types 1 and 2. The original fields are

$$\phi^a(x) = \int \frac{d^3\boldsymbol{p}}{\sqrt{(2\pi)^3}\sqrt{2\omega_{\boldsymbol{p}}}} \left(a_{\boldsymbol{p}}^{(a)} e^{-ip\cdot x} + a_{\boldsymbol{p}}^{(a)\dagger} e^{ip\cdot x}\right) \tag{10}$$

and Coleman now considers the fields

$$\psi = \frac{1}{\sqrt{2}}\left(\phi^1 + i\phi^2\right) \tag{11}$$

$$\psi^* = \frac{1}{\sqrt{2}}\left(\phi^1 - i\phi^2\right) \tag{12}$$

The original fields 10 are real functions, since the two terms in the parentheses are complex (or hermitian) conjugates of each other. Thus the new fields ψ and ψ^* are complex fields. They can be written as Fourier integrals over the new operators as

$$\psi(x) = \int \frac{d^3\boldsymbol{p}}{\sqrt{(2\pi)^3}\sqrt{2\omega_{\boldsymbol{p}}}} \left(b_{\boldsymbol{p}} e^{-ip\cdot x} + c_{\boldsymbol{p}}^\dagger e^{ip\cdot x}\right) \tag{13}$$

$$\psi^\dagger(x) = \int \frac{d^3\boldsymbol{p}}{\sqrt{(2\pi)^3}\sqrt{2\omega_{\boldsymbol{p}}}} \left(b_{\boldsymbol{p}}^\dagger e^{ip\cdot x} + c_{\boldsymbol{p}} e^{-ip\cdot x}\right) \tag{14}$$

Coleman goes through the derivations and shows that we get quite neat results, such as

$$[Q,\psi] = -\psi \tag{15}$$

$$[Q,\psi^*] = \psi^* \tag{16}$$

The Lagrangian 1 turns out to be

$$\mathscr{L} = \partial_\mu\psi^*\partial^\mu\psi - \mu^2\psi^*\psi \tag{17}$$

Coleman then proceeds to treat ψ and ψ^* as independent fields, as is done by most other books. The difference is that Coleman actually gives some justification for doing this, which is a point that is sadly lacking in every other book I've looked at. I did make an earlier attempt myself to justify this, but Coleman's treatment is quite different so it's worth looking at here.

We consider a general Lagrangian of the form

$$\mathscr{L}\left(\psi,\psi^*,\partial^\mu\psi,\partial^\mu\psi^*\right) \tag{18}$$

The action integral is then

$$\mathcal{S}=\int d^{4}x\ \mathscr{L} \tag{19}$$

Applying the principle of least action, we need to vary the action over some fixed time interval and require that the variation is zero. Thus we have

$$\delta\mathcal{S}=\int d^{4}x\ \left(A\delta\psi+B\delta\psi^{*}\right)=0 \tag{20}$$

where

$$A=\frac{\partial\mathscr{L}}{\partial\psi} \tag{21}$$

and thus is some function of $(\psi,\psi^{*},\partial^{\mu}\psi,\partial^{\mu}\psi^{*})$. The precise form of this function isn't important for the proof.

Coleman takes $B=A^{*}$, but I don't think this is required, since in general, the Lagrangian 18 need not be symmetric with respect to the field and its complex conjugate (although the example 17 that we used above is symmetric this way). In any case, it's not required that $B=A^{*}$ for Coleman's proof to work.

If we were treating ψ and ψ^{*} as independent variables, then in principle we should be able to vary each of them independently of the other. That is we should be able to take $\delta\psi$ to be some arbitrary function while taking $\delta\psi^{*}$ to be zero. However, since $\delta\psi^{*}$ is the complex conjugate of $\delta\psi$, we clearly cannot do this. However, if we *did* do this, then we'd have

$$\delta\mathcal{S}=\int d^{4}x\ A\delta\psi=0 \tag{22}$$

and, since $\delta\psi$ is arbitrary, we'd therefore require

$$A=0 \tag{23}$$

Remember from 21 that A is a function, not just a constant, so requiring $A=0$ gives us the equations of motion in the usual way.

By the same token, if we took $\delta\psi^{*}$ to be some arbitrary function and took $\delta\psi=0$, we'd get the condition

$$B=0 \tag{24}$$

which would give us another equation of motion, since $B=B\left(\psi,\psi^{*},\partial^{\mu}\psi,\partial^{\mu}\psi^{*}\right)$.

Again, though, this isn't legitimate - we can't vary $\delta\psi$ and $\delta\psi^{*}$ completely independently of each other. However, it *is* allowed to take $\delta\psi$ to be an arbitrary function and then require $\delta\psi=\delta\psi^{*}$. This restricts $\delta\psi$ to be a real function, and satisfies the condition that $\delta\psi$ and $\delta\psi^*$ are complex conjugates of each other. In this case, from 20, we get

$$\delta\mathcal{S} = \int d^4x\ (A+B)\,\delta\psi = 0 \tag{25}$$

which in turn gives us the equation of motion in the form

$$A+B=0 \tag{26}$$

Likewise, we could take $\delta\psi$ to be an arbitrary function and then require $\delta\psi = -\delta\psi^*$. This means that $\delta\psi$ is purely imaginary, but still satisfies the condition that $\delta\psi$ and $\delta\psi^*$ are complex conjugates of each other. In this case

$$\delta\mathcal{S} = \int d^4x\ (A-B)\,\delta\psi = 0 \tag{27}$$

and another equation of motion is

$$A-B=0 \tag{28}$$

Combining this with the first equation of motion 26 we find

$$A=B=0 \tag{29}$$

That is, we get the same result from this line of reasoning as we did by treating $\delta\psi$ and $\delta\psi^*$ *as if* they were *completely* independent of each other. Thus although it's not mathematically correct to assume ψ and ψ^* are independent, we still get the correct equations of motion if we *do* assume this.

This technique works, however, only in the case where we have two fields. If we have a Lagrangian with more than two fields, things get considerably messier, although in some cases we can pick out two of these fields and combine them to create a complex field and its conjugate. These two fields then become a subset of the total number of fields.

PINGBACKS

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