CHARGE CONJUGATION

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Reference: Sidney Coleman, *Quantum Field Theory: Lectures of Sidney Coleman* (edited by Bryan Gin-ge Chen *et al.*), World Scientific, 2019. Section 6.3.

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Coleman's initial treatment of charge conjugation is for a pair of scalar fields, which is a simpler case than that we looked at earlier. We look again at the Lagrangian for two scalar fields

$$\mathscr{L}=\frac{1}{2}\left(\partial^{\mu}\phi^{a}\partial_{\mu}\phi^{a}-\mu^{2}\phi^{a}\phi^{a}\right) \tag{1}$$

where $a=1,2$ so there are two distinct fields. We saw that this Lagrangian is invariant under the continuous transformation

$$\phi^{1}\to\phi^{1}\cos\lambda+\phi^{2}\sin\lambda \tag{2}$$

$$\phi^{2}\to\phi^{2}\cos\lambda-\phi^{1}\sin\lambda \tag{3}$$

for some parameter λ. This is effectively a rotation in the space spanned by the two fields. This rotation is not a rotation in actual spacetime; it's a rotation in the abstract space defined by the field operators. This invariance is known as SO(2) invariance, which is invaraince under the Special Orthogonal (SO) group in 2 dimensions.

Drawing an analogy with 2-dimensional space, we can view the rotation as a rotation in the space with axes defined by ϕ^{1} and ϕ^{2}. We can also define a reflection in this space. In 2-dimensional space, a reflection in one of the coordinate axes (say the x axis) is obtained by mapping all y coordinates to $-y$ and leaving the x values unchanged. In our ϕspace we can define a reflection by mapping ϕ^{2} to $-\phi^{2}$ and leaving ϕ^{1} unchanged. That is, we have

$$\phi^{1}\to\phi^{1} \tag{4}$$

$$\phi^{2}\to-\phi^{2} \tag{5}$$

From 1, we see that this reflection leaves the Lagrangian unchanged, since ϕ^{a} and its derivative always appear in squared terms, so the minus sign cancels out.

The difference between the reflection transformation and the rotation transformation is that reflection is a *discrete* transformation, in the sense that it is all or nothing. You can either reflect the fields or not reflect them; you can't reflect them by some fraction such as 10%. Therefore, there is no λ parameter involved in a reflection, so we can't apply Noether's theorem to obtain a conserved current.

We can define a unitary operator U to implement this reflection by requiring

$$U^{\dagger}\phi^{1}U = \phi^{1} \tag{6}$$

$$U^{\dagger}\phi^{2}U = -\phi^{2} \tag{7}$$

For a free state (no interactions) with definite numbers of types 1 and 2 particles is

$$U = (-1)^{N_2} \tag{8}$$

where N_2 is the number of type 2 particles. This operator is real as is its own inverse (applying it twice just gives the unit operator), so

$$U^{\dagger} = U^{-1} = U \tag{9}$$

It satisfies 6 since N_2 commutes with ϕ^1 so

$$U^{\dagger}\phi^{1}U = (-1)^{N_2}\phi^{1}(-1)^{N_2} \tag{10}$$

$$= (-1)^{2N_2}\phi^{1} \tag{11}$$

$$= \phi^{1} \tag{12}$$

For ϕ^2, the field operator ϕ^2 either creates or annihilates a type 2 particle so

$$U^{\dagger}\phi^{2}U = (-1)^{N_2\pm 1}\phi^{2}(-1)^{N_2} \tag{13}$$

$$= (-1)^{2N_2\pm 1}\phi^{2} \tag{14}$$

$$= -\phi^{2} \tag{15}$$

When we expand the fields in terms of their creation and annihilation operators, we have

$$\phi^{a}(x) = \int \frac{d^{3}\boldsymbol{p}}{\sqrt{(2\pi)^{3}}\sqrt{2\omega_{\boldsymbol{p}}}}\left(a_{\boldsymbol{p}}^{(a)}e^{-ip\cdot x} + a_{\boldsymbol{p}}^{(a)\dagger}e^{ip\cdot x}\right) \tag{16}$$

The effect of U on these operators must then be (using the same reasoning)

$$U^{\dagger}a_{\boldsymbol{p}}^{(1)}U = a_{\boldsymbol{p}}^{(1)} \tag{17}$$

$$U^{\dagger}a_{\boldsymbol{p}}^{(2)}U = -a_{\boldsymbol{p}}^{(2)} \tag{18}$$

$$U^{\dagger}a_{\boldsymbol{p}}^{(1)\dagger}U = a_{\boldsymbol{p}}^{(1)\dagger} \tag{19}$$

$$U^{\dagger}a_{\boldsymbol{p}}^{(2)\dagger}U = -a_{\boldsymbol{p}}^{(2)\dagger} \tag{20}$$

When considering 1, we defined the auxiliary operators

$$b_{\boldsymbol{p}} = \frac{1}{\sqrt{2}}\left(a_{\boldsymbol{p}}^{(1)} + ia_{\boldsymbol{p}}^{(2)}\right) \tag{21}$$

$$b_{\boldsymbol{p}}^{\dagger} = \frac{1}{\sqrt{2}}\left(a_{\boldsymbol{p}}^{(1)\dagger} - ia_{\boldsymbol{p}}^{(2)\dagger}\right) \tag{22}$$

$$c_{\boldsymbol{p}} = \frac{1}{\sqrt{2}}\left(a_{\boldsymbol{p}}^{(1)} - ia_{\boldsymbol{p}}^{(2)}\right) \tag{23}$$

$$c_{\boldsymbol{p}}^{\dagger} = \frac{1}{\sqrt{2}}\left(a_{\boldsymbol{p}}^{(1)\dagger} + ia_{\boldsymbol{p}}^{(2)\dagger}\right) \tag{24}$$

Coleman shows that under the unitary transformaton 6, $b_{\boldsymbol{p}}$ swaps with $c_{\boldsymbol{p}}$, so the reflection operation swaps these two types of particles. This has the effect that for

$$Q = \int d^3\boldsymbol{p}\ \left[b_{\boldsymbol{p}}^{\dagger}b_{\boldsymbol{p}} - c_{\boldsymbol{p}}^{\dagger}c_{\boldsymbol{p}}\right] \tag{25}$$

$$= N_b - N_c \tag{26}$$

Q gets mapped into $-Q$. If we interpret Q as the net charge on a collection of b and c type particles, then U swaps negative and positive charges. For this reason, it's known as *charge conjugation*. As Coleman points out, 'conjugation' isn't the best term for this transformation, as it brings to mind complex conjugation which is not a unitary operation.