TIME REVERSAL

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Reference: Sidney Coleman, *Quantum Field Theory: Lectures of Sidney Coleman* (edited by Bryan Gin-ge Chen *et al.*), World Scientific, 2019. Section 6.3.

Post date: 6 Feb 2020.

We've covered time reversal and anti-unitary operators before, so we'll just add a few notes on Coleman's treatment of the subject.

First, why is it necessary that the time-reversal operator be anti-unitary instead of unitary? If we have a system described by a position $q(t)$ as a function of time and a momentum $p(t)$ also as a function of time, then reversing time changes t to $-t$ and also reverses the direction of the momentum, but leaves the position unchanged. Thus

$$q(t) \to q(-t) \tag{1}$$

$$p(t) \to -p(-t) \tag{2}$$

Suppose we have a unitary operator U_T that accomplishes this, so that

$$U_T^\dagger q(t) U_T = q(-t) \tag{3}$$

$$U_T^\dagger p(t) U_T = -p(-t) \tag{4}$$

Now consider the commutator

$$[q(t), p(t)] = i \tag{5}$$

Time-reversing this and evaluating at $t=0$ gives us Coleman's eqn 6.102

$$U_T^\dagger [q(t), p(t)] U_T = -i \tag{6}$$

That is, the time reversed commutator gives the negative of the original.

Also, if we consider the propagator e^{-iHt} where H is the hamiltonian, then the time reversed version should give a propagator in the opposite direction. That is

$$U_T^\dagger e^{-iHt} U_T = e^{iHt} \tag{7}$$

However, if we take the time derivative of this equation we get

$$U_T^\dagger(-iH)e^{-iHt}U_T = iHe^{iHt} \tag{8}$$

At $t = 0$ this is

$$U_T^\dagger(-iH)U_T = iH \tag{9}$$

Since a unitary operator has no effect on i (it's just a number), we can cancel it and find that

$$U_T^\dagger H U_t = -H \tag{10}$$

Two operators related by a unitary transformation have the same spectrum of eigenvalues, which in this case, means the same energy levels. However, if H and $-H$ both have the same eigenvalues, there must be a negative eigenvalue that is equal and opposite to every positive eigenvalue, which means we can't find a lower bound for the energy. This means that there is no ground state.

These problems are resolved by defining the time reversal operator to be anti-unitary, with the symbol Ω. Coleman gives a detailed description of the properties of anti-unitary operators and shows how the two apparent contradictions above can be resolved by using them. The important point is that any anti-unitary operator can be written as the product of a unitary operator U and the complex conjugation operator K. That is

$$\Omega = UK \tag{11}$$

We can verify this by taking $U = \Omega K$ and using the fact that $K = K^{-1}$ (applying complex conjugation twice is equivalent to the unit operator):

$$\Omega = (\Omega K)K = \Omega K^2 = \Omega \tag{12}$$

Coleman concludes by considering the effect of time reversal on a scalar field ϕ. He states that the formulas defining the creation and annihilation operators $a_{\boldsymbol{p}}^\dagger$ and $a_{\boldsymbol{p}}$ involve only real numbers. I'm not quite sure which formulas he's referring to, but if we go back to their original definition as raising and lowering operators for the harmonic oscillator (Coleman's eqn 2.17)

These 2 formulas are from Griffiths's QM book.

$$a_+ = \frac{1}{\sqrt{2\hbar m\omega}}[-ip + m\omega x] \tag{13}$$

$$a_- = \frac{1}{\sqrt{2\hbar m\omega}}[ip + m\omega x] \tag{14}$$

we can see that, in the position representation, $p=-i\hbar\partial_x$ so both a_+ and a_- are real. The extension of these operators to field theory is (I think) done by postulating the commutation relations in Coleman's eqn 2.47:

$$\left[a_{\boldsymbol{p}},a_{\boldsymbol{p}'}\right]=\left[a_{\boldsymbol{p}}^{\dagger},a_{\boldsymbol{p}'}^{\dagger}\right]=0 \tag{15}$$

$$\left[a_{\boldsymbol{p}},a_{\boldsymbol{p}'}^{\dagger}\right]=\delta^{(3)}\left(\boldsymbol{p}-\boldsymbol{p}'\right) \tag{16}$$

These formulas also involve only real numbers.

Coleman considers the combined operator Ω_{PT} which applies both parity and time reversal, so that all four components of spacetime are reversed. This treats all four components equally and is thus works well with Lorentz transformations. Combining parity and time reversal reverses the momentum $\boldsymbol{p}$ twice, thus leaving it unchanged. Since $a_{\boldsymbol{p}}$ and $a_{\boldsymbol{p}}^{\dagger}$ involve only real numbers, Ω_{PT} has no effect on them, and thus commutes with them so we have

$$\Omega_{PT}a_{\boldsymbol{p}}=a_{\boldsymbol{p}}\Omega_{PT} \tag{17}$$

$$\Omega_{PT}a_{\boldsymbol{p}}^{\dagger}=a_{\boldsymbol{p}}^{\dagger}\Omega_{PT} \tag{18}$$

or

$$a_{\boldsymbol{p}}=\Omega_{PT}^{-1}a_{\boldsymbol{p}}\Omega_{PT} \tag{19}$$

$$a_{\boldsymbol{p}}^{\dagger}=\Omega_{PT}^{-1}a_{\boldsymbol{p}}^{\dagger}\Omega_{PT} \tag{20}$$

At this stage, there is no difference between Ω_{PT} and the unit operator: both leave $a_{\boldsymbol{p}}$ and $a_{\boldsymbol{p}}^{\dagger}$ unchanged. However, if we consider the expansion of the scalar field

$$\phi(x)=\int\frac{d^3\boldsymbol{p}}{\sqrt{(2\pi)^3}\sqrt{2\omega_{\boldsymbol{p}}}}\left(a_{\boldsymbol{p}}e^{-ip\cdot x}+a_{\boldsymbol{p}}^{\dagger}e^{ip\cdot x}\right) \tag{21}$$

we see that applying Ω_{PT} gives us

$$\Omega_{PT}^{-1}\phi(x)\Omega_{PT}=\Omega_{PT}^{-1}\int\frac{d^3\boldsymbol{p}}{\sqrt{(2\pi)^3}\sqrt{2\omega_{\boldsymbol{p}}}}\left(a_{\boldsymbol{p}}e^{-ip\cdot x}+a_{\boldsymbol{p}}^{\dagger}e^{ip\cdot x}\right)\Omega_{PT} \tag{22}$$

Parity and time reversal leave $\boldsymbol{p}$ unchanged, so this operation has no effect on the $\frac{d^3\boldsymbol{p}}{\sqrt{(2\pi)^3}\sqrt{2\omega_{\boldsymbol{p}}}}$ factor. It also has no effect on $a_{\boldsymbol{p}}$ and $a_{\boldsymbol{p}}^{\dagger}$ so the only effect is on the two exponentials. The product $p\cdot x$ is real (remember that

p and x are four-vectors), so the only effect on the exponent is to take the complex conjugate, swapping $i \leftrightarrow -i$, giving

$$\Omega_{PT}^{-1}\phi(x)\,\Omega_{PT} = \int \frac{d^3\boldsymbol{p}}{\sqrt{(2\pi)^3}\sqrt{2\omega_{\boldsymbol{p}}}}\left(a_{\boldsymbol{p}}e^{ip\cdot x} + a_{\boldsymbol{p}}^{\dagger}e^{-ip\cdot x}\right) \tag{23}$$

Thus the net effect is to map $\phi(x) \rightarrow \phi(-x)$, which is the desired effect of the combined parity-time reversal operation.