SCHRÖDINGER, HEISENBERG AND INTERACTION PICTURES

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Reference: Sidney Coleman, *Quantum Field Theory: Lectures of Sidney Coleman* (edited by Bryan Gin-ge Chen *et al.*), World Scientific, 2019. Sections 7.1 - 7.2.

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Although we've used these three pictures of quantum mechanics in various places, it's good to collect a summary of them in one place.

The Schrödinger picture is the one in which all the time dependence is placed in the quantum states $|\psi(t)\rangle$ and not in the operators. The fundamental operators, position q and momentum p are not time-dependent, so we can write them as q_S and p_S (subscript S for Schrödinger) without any time argument. The states obey the standard Schrödinger equation

$$i\frac{d}{dt}|\psi(t)\rangle = H(p_S, q_S, t)|\psi(t)\rangle \tag{1}$$

The Hamiltonian may have an explicit time dependence, but if it doesn't, it doesn't have any indirect time dependence since the operators p_S and q_S are time-independent.

The evolution of a Schrödinger state is obtained from a time evolution operator $U(t,t')$ according to

$$|\psi(t)\rangle = U\left(t,t'\right)\left|\psi\left(t'\right)\right\rangle \tag{2}$$

This operator take a state at time t' and evolves it to another time t (t could be earlier than, later than or equal to t'). U is a linear operator because the Schrödinger equation is linear, so the state $|\psi(t)\rangle$ must obey the Schrödinger equation at all times. Because the Schrödinger equation conserves probability, we must have, assuming the state is normalized:

$$\langle\psi(t)|\psi(t)\rangle = 1 \tag{3}$$

for all times, so we must have

$$\left\langle U\left(t,t'\right)\psi\left(t'\right)\middle|U\left(t,t'\right)\psi\left(t'\right)\right\rangle = \left\langle\psi\left(t'\right)U^{\dagger}\left(t,t'\right)\middle|U\left(t,t'\right)\psi\left(t'\right)\right\rangle \tag{4}$$

$$= \left\langle\psi\left(t'\right)\middle|\psi\left(t'\right)\right\rangle \tag{5}$$

The equality of these two lines is a requirement, not a derivation, so we can deduce that we must have

$$U^{\dagger}\left(t,t'\right)=U^{-1}\left(t,t'\right) \tag{6}$$

which means the operator U is unitary. There is also a composition rule:

$$U\left(t,t''\right)=U\left(t,t'\right)U\left(t',t''\right) \tag{7}$$

Combining this with 6 we have

$$U\left(t',t\right)=U^{-1}\left(t,t'\right) \tag{8}$$

A differential equation for U can be obtained by differentiating 2 and using 1. First we differentiate 2

$$i\frac{d}{dt}\left|\psi\left(t\right)\right\rangle=i\frac{\partial}{\partial t}U\left(t,t'\right)\left|\psi\left(t'\right)\right\rangle \tag{9}$$

Next we have from 1

$$i\frac{d}{dt}\left|\psi\left(t\right)\right\rangle = H\left(p_S,q_S,t\right)\left|\psi\left(t\right)\right\rangle \tag{10}$$

$$= H\left(p_S,q_S,t\right)U\left(t,t'\right)\left|\psi\left(t'\right)\right\rangle \tag{11}$$

Comparing 9 and 11 we have

$$i\frac{\partial}{\partial t}U\left(t,t'\right)=H\left(p_S,q_S,t\right)U\left(t,t'\right) \tag{12}$$

with the initial condition

$$U\left(t',t'\right)=1 \tag{13}$$

This is just saying that if we evolve a state through zero time, nothing changes. If H doesn't depend explicitly on t, a formal solution of 12 is

$$U\left(t,t'\right)=e^{-iH\left(p_S,q_S\right)\left(t-t'\right)} \tag{14}$$

In the Heisenberg picture, the states are time-independent, with all the time dependence being shifted to the operators. That is, we can write

$$\left|\psi\left(t\right)\right\rangle_H=\left|\psi\left(0\right)\right\rangle_H \tag{15}$$

for all t, so in fact we can just drop the t dependence from a Heisenberg state. Usually the Heisenberg state is taken to be the Schrödinger state at $t=0$, so that

$$\left|\psi\right\rangle_H=\left|\psi\left(0\right)\right\rangle_S \tag{16}$$

If we start with a Schrödinger state at some time t, we can apply 14 in reverse to de-evolve the state back to $t = 0$, so we have

$$|\psi\rangle_H = e^{iH(p_S,q_S)t}|\psi(t)\rangle_S \tag{17}$$

The fundamental position and momentum operators are now defined to be time dependent. In the Schrödinger picture, the expectation value of q_S is obtained from the square modulus of $\langle\psi(t)|q_S|\psi(t)\rangle$. If we want to transfer the time dependence to the operator, we have

$$_S\langle\psi(t)|q_S|\psi(t)\rangle_S = {}_S\langle U(t,0)\psi(0)|q_S|U(t,0)\psi(0)\rangle_S \tag{18}$$

$$= {}_S\left\langle\psi(0)\middle|U^\dagger(t,0)\,q_S U(t,0)\middle|\psi(0)\right\rangle_S \tag{19}$$

$$= {}_H\left\langle\psi\middle|U^\dagger(t,0)\,q_S U(t,0)\middle|\psi\right\rangle_H \tag{20}$$

where we used 16 to get the last line. We can now identify the operator in the middle with the Heisenberg position operator, which is now a function of time:

$$q_H(t) = U^\dagger(t,0)\,q_S U(t,0) \tag{21}$$

A similar equation holds for p_H:

$$p_H(t) = U^\dagger(t,0)\,p_S U(t,0) \tag{22}$$

The Heisenberg and Schrödinger operators are equal at $t = 0$:

$$q_H(0) = q_S \tag{23}$$

$$p_H(0) = p_S \tag{24}$$

We can follow a similar procedure to convert any operator from the Schrödinger to the Heisenberg picture, as Coleman does in eqn 7.15. Note that a general operator $A_S(t)$ *may* have an explicit time dependence (for example, a Hamiltonian $H(p_S,q_S,t)$), so the general rule is

$$A_H(t) = U^\dagger(t,0)\,A_S(t)\,U(t,0) \tag{25}$$

$$= U(0,t)\,A_S(t)\,U^\dagger(0,t) \tag{26}$$

where the last line follows from 8.

The interaction picture (also called the Dirac picture) is the prelude to perturbation theory. We assume that we have a free time-independent Hamiltonian H_0 and an extra bit H' that could depend on time. That is

$$H = H_0(p,q) + H'(p,q,t) \tag{27}$$

In the interaction picture, we define an interaction version of q by using 21, but where the evolution operator U depends *only* on H_0 (so we'll call it U_0):

$$q_I(t) = U_0^\dagger(t,0)\, q_S U_0(t,0) \tag{28}$$

$$= e^{iH_0(p_S,q_S)t} q_S e^{-iH_0(p_S,q_S)t} \tag{29}$$

We also define interaction picture states as

$$|\psi(t)\rangle_I = e^{iH_0(p_S,q_S)t}|\psi(t)\rangle_S \tag{30}$$

and a general transformation for operators

$$A_I(t) = e^{iH_0(p_S,q_S)t} A_S(t)\, e^{-iH_0(p_S,q_S)t} \tag{31}$$

Note that these definitions conserve probability and expectation values, as in

$${}_I\langle\psi(t)|A_I(t)|\psi(t)\rangle_I = {}_S\left\langle\psi(t)\left|e^{-iH_0(p_S,q_S)t}e^{iH_0(p_S,q_S)t}A_S(t)\,e^{-iH_0(p_S,q_S)t}e^{iH_0(p_S,q_S)t}\right|\psi(t)\right\rangle_S \tag{32}$$

$$= {}_S\langle\psi(t)|A_S(t)|\psi(t)\rangle_S \tag{33}$$

The main goal of the interaction picture is to derive a differential equation for the interaction state $|\psi(t)\rangle_I$ which in general will not be exactly solvable. We then apply perturbation theory to find an approximate solution. We obtain the differential equation from 30:

$$\frac{d}{dt}|\psi(t)\rangle_I = e^{iH_0(p_S,q_S)t}\left[iH_0(p_S,q_S)|\psi(t)\rangle_S + \frac{d}{dt}|\psi(t)\rangle_S\right] \tag{34}$$

The last derivative on the RHS is obtained by applying the Schrödinger equation 1, but we need to remember that the H in 1 is the *full* Hamiltonian H 27 and not just the free Hamiltonian H_0 that we've been using so far. Using this, Coleman shows in eqn 7.22 and 7.23 that the differential equation is

$$\frac{d}{dt}|\psi(t)\rangle_I = -iH'(p_I,q_I,t)|\psi(t)\rangle_I \tag{35}$$

where

$$H'(p_I,q_I,t) \equiv H_I(t) = e^{iH_0(p_S,q_S)t}H'_S(p_S,q_S,t)\,e^{-iH_0(p_S,q_S)t} \tag{36}$$

Note also that the position and momentum operators have morphed from (p_S, q_S) in 34 to (p_I, q_I) in 35. This is possible if we assume that the Hamiltonian $H'_S(p_S, q_S, t)$ in 36 can be expanded in a power series in p_S and q_S, so we can then insert factors of $e^{iH_0(p_S,q_S)t}e^{-iH_0(p_S,q_S)t}$ between all pairs of factors of p_S and q_S to conver them to p_I and q_I using 29.

To use perturbation theory, an interaction picture evolution operator $U_I(t,t')$ is introduced by the equation

$$|\psi(t)\rangle_I = U_I(t,t')\left|\psi(t')\right\rangle_I \tag{37}$$

This evolution operator has the same properties as the original evolution operator introduced in 2. The relation between U_I and the original U (the latter of which depends on the full Hamiltonian H) is derived in Coleman's eqn 7.31:

$$U_I(t,0) = e^{iH_0t}U(t,0) = e^{iH_0t}e^{-iHt} \tag{38}$$

Note carefully that the first exponent has the free Hamiltonian H_0 while the second has the full Hamiltonian H.

Coleman develops the properties of U_I in eqns 7.25 through 7.33, ending with the central differential equation for U_I which is to be solved using perturbation theory:

$$\boxed{i\frac{\partial}{\partial t}U_I(t,t') = H_I(t)U_I(t,t')} \tag{39}$$

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