DYSON'S FORMULA AND TIME ORDERING

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Reference: Sidney Coleman, *Quantum Field Theory: Lectures of Sidney Coleman* (edited by Bryan Gin-ge Chen *et al.*), World Scientific, 2019. Section 7.3.

Post date: 20 Feb 2020.

The differential equation that needs to be solved to obtain the evolution operator U_I in the interaction picture is

$$i\frac{\partial}{\partial t}U_I\left(t,t'\right)=H_I\left(t\right)U_I\left(t,t'\right) \tag{1}$$

where H_I is the interaction Hamiltonian. In most cases, H_I is not a one-dimensional function, so the simple solution

$$U_I\left(t,t'\right)=\exp\left(-i\int_{t'}^{t}dt''H_I\left(t''\right)\right) \tag{2}$$

won't work. The problem is that the exponential of an integral of an operator contains commutators of the operator at different times and, since in general, H_I doesn't commute with itself at different times, we can't just do the integral and then take the exponential.

The problem was solved, in a formal way (not terribly conducive to calculation), by Freeman Dyson in 1949. The solution is *Dyson's formula*:

$$U_I\left(t,t'\right)=T\exp\left(-i\int_{t'}^{t}dt''H_I\left(t''\right)\right) \tag{3}$$

where the T in front of the exponential is the *time-ordering* symbol. It is interpreted as ordering all terms in decreasing order of time, from left to right. As Coleman puts it, 'left-ist is latest' or 'later on the left'. The time ordering procedure is always to be done first, before any products of terms.

To see what this means when applied to 3, we need to express this formula so that it contains products of terms, which means we have to expand it in a power series. This done in Coleman's eqn 7.37. The first two terms are

$$U_I\left(t,t'\right)=T\left(1-i\int_{t'}^{t}dt_1\;H_I\left(t_1\right)+\ldots\right) \tag{4}$$

Since these two terms each consist of only a single factor, time ordering has no effect. The effect first becomes visible on the next term:

$$U_I\left(t,t'\right)=1-i\int_{t'}^{t}dt_1\;H_I\left(t_1\right)+\frac{\left(-i\right)^2}{2!}T\left[\int_{t'}^{t}\int_{t'}^{t}dt_1dt_2H_I\left(t_1\right)H_I\left(t_2\right)\right]+\ldots \tag{5}$$

The double integral contains portions where $t_1<t_2$ and also portions where $t_1>t_2$, since both integration variables extend over the same range. The effect of the time ordering can be broken down into two terms:

$$T\left[\int_{t'}^{t}\int_{t'}^{t}dt_1dt_2H_I\left(t_1\right)H_I\left(t_2\right)\right]=\int_{t'}^{t}dt_2\left[H_I\left(t_2\right)\int_{t'}^{t_2}dt_1H_I\left(t_1\right)\right]+\int_{t'}^{t}dt_1\left[H_I\left(t_1\right)\int_{t'}^{t_1}dt_2H_I\left(t_2\right)\right] \tag{6}$$

In the first term on the RHS, t_2 is always later than t_1, while the opposite is true in the second term. In fact, since t_1 and t_2 are just dummy integration variables, these two terms are the same, so we could write

$$T\left[\int_{t'}^{t}\int_{t'}^{t}dt_1dt_2H_I\left(t_1\right)H_I\left(t_2\right)\right]=2\int_{t'}^{t}dt_2\left[H_I\left(t_2\right)\int_{t'}^{t_2}dt_1H_I\left(t_1\right)\right] \tag{7}$$

That this formula provides a solution to 1 can be verified directly. Taking the derivative of both sides with respect to t (not t'), we have

$$i\frac{\partial}{\partial t}U_I\left(t,t'\right)=iT\exp\left(-iH_I\left(t\right)\int_{t'}^{t}dt''H_I\left(t''\right)\right) \tag{8}$$

Now provided $t>t'$, the time ordering will always place the $H_I\left(t\right)$ term on the left, so it can be taken outside the time-ordered term, giving

$$i\frac{\partial}{\partial t}U_I\left(t,t'\right)=iT\exp\left(\left(-iH_I\left(t\right)\right)\left(-i\int_{t'}^{t}dt''H_I\left(t''\right)\right)\right) \tag{9}$$

$$=i\left(-iH_I\left(t\right)\right)T\exp\left(-iH_I\left(t\right)\int_{t'}^{t}dt''H_I\left(t''\right)\right) \tag{10}$$

$$=H_I\left(t\right)T\exp\left(-iH_I\left(t\right)\int_{t'}^{t}dt''H_I\left(t''\right)\right) \tag{11}$$

which agrees with 1.

PINGBACKS

Pingback: Wick diagrams