SCATTERING AND THE S-MATRIX

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Reference: Sidney Coleman, *Quantum Field Theory: Lectures of Sidney Coleman* (edited by Bryan Gin-ge Chen *et al.*), World Scientific, 2019. Section 7.4.

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In section 7.4, Coleman considers in a general way the problem of how to calculate the probability of scattering from an initial state to a final state. There are a couple of points that aren't quite clear to me here, so any comments would be welcome.

The idea is that we start with a system in which the particles that are to scatter are widely separated in time and space, so that they can be treated essentially as free wave packets. Note that it's important to consider a particle as a wave *packet* (which can be normalized, since it's assumed that its amplitude falls off rapidly enough as we move far from the centre of the packet) and not as a plane wave, which has the same amplitude everywhere and thus cannot be normalized.

We consider a free wave packet to be represented by the ket $|\psi(t)\rangle$ in the Schrödinger picture (where the time dependence is in the states and not the operators). This state evolves according to the free Hamiltonian H_0:

$$|\psi(t)\rangle = e^{-iH_0t}|\psi\rangle \tag{1}$$

where a state such as $|\psi\rangle$ with no explicit time dependence is defined to be the state at $t=0$.

In a scattering experiment, the Hamiltonian H must have a potential term which describes the interaction. The assumption is that, for $t\to-\infty$, there is a state which obeys this interacting Hamiltonian which is indistinguishable from the free state. We call this exact state $|\psi(t)\rangle^{\text{in}}$. This state evolves according to

$$|\psi(t)\rangle^{\text{in}} = e^{-iHt}|\psi\rangle^{\text{in}} \tag{2}$$

The equivalence of the exact and free states in the distant past is given by the condition

$$\lim_{t\to-\infty}\left\| e^{-iH_0t}|\psi\rangle - e^{-iHt}|\psi\rangle^{\text{in}}\right\| = 0 \tag{3}$$

Similarly, after the scattering has occurred, the products of the interaction will fly apart and ultimately become so distant from each other that they can again be treated as free wave packets. We define the state $|\phi(t)\rangle$ to be the state consisting of free wave packets and $|\phi(t)\rangle^{\text{out}}$ to be the exact state after scattering. Again, these two states should become asymptotically equal for large time, so we have

$$\lim_{t\to\infty}\left\|e^{-iH_0t}|\phi\rangle-e^{-iHt}|\phi\rangle^{\text{out}}\right\|=0 \tag{4}$$

In a scattering experiment, we construct the incoming state $|\psi(t)\rangle^{\text{in}}$ and measure the outgoing state $|\phi(t)\rangle^{\text{out}}$. The amplitude for scattering between these two states is then ${}^{\text{out}}\langle\phi(t)|\psi(t)\rangle^{\text{in}}$. Coleman says that we actually don't need to include the time dependence in this amplitude since both states $|\psi(t)\rangle^{\text{in}}$ and $|\phi(t)\rangle^{\text{out}}$ evolve according to the exact Hamiltonian. That is, we have

$$
\begin{aligned}
{}^{\text{out}}\langle\phi(t)|\psi(t)\rangle^{\text{in}} &= {}^{\text{out}}\left\langle\phi e^{iHt}\middle|e^{-iHt}\psi\right\rangle^{\text{in}} && (5)\\
&= {}^{\text{out}}\langle\phi|\psi\rangle^{\text{in}} && (6)
\end{aligned}
$$

This is OK provided we start out with both states at the same time and apply the evolution operator e^{-iHt} to both of states. I suspect this is valid since both states $|\psi(t)\rangle^{\text{in}}$ and $|\phi(t)\rangle^{\text{out}}$ are solutions of the same Schrödinger equation with the exact Hamiltonian, so they are both valid at all times, before, during and after the interaction. It's just that in the lab, we would only encounter $|\psi(t)\rangle^{\text{in}}$ before scattering and $|\phi(t)\rangle^{\text{out}}$ after scattering.

The S-matrix is then defined so that

$$\langle\phi|\mathrm{S}|\psi\rangle={}^{\text{out}}\langle\phi|\psi\rangle^{\text{in}} \tag{7}$$

As an illustration of this situation, Coleman describes a system of 3 incoming particles, where this system can exist in one of two states. Two of the particles can be in a bound state with the third particle free, or else all three particles can be free. The scattering process could have several outcomes. One possibility is that the first incoming state (a bound state of two particles plus a free third one) could simply scatter off each other and emerge in the same state. Another possibility is that the bound state could be ionized by the scattering so that the outgoing state consists of 3 separate particles, and so on. Coleman states that the two possibilities for the incoming state must be orthogonal, since the system can't be in both states at once. That is, a system that looks like one free particle and one bound state can't also look like three free particles. Given that quantum mechanics allows a state that is a linear combination of other states, I'm not entirely sure how we can assert this. Indeed, one of the main predictions of quantum mechanics is that such superpositions of states *do* exist and that the wave function collapses to a single state only upon measurement.

Anyway, to carry on. Coleman now introduces a damping function $f(t,T,\Delta)$ which slowly (over a time span Δ) turns the interaction on at some finite time in the past, maintains it at full strength for a time T and then slowly turns it off again, also with a time span Δ. With such a system, we can now say that the two free states $|\psi\rangle$ and $|\phi\rangle$ are identical to the exact states $|\psi(t)\rangle^{\text{in}}$ and $|\phi(t)\rangle^{\text{out}}$. Coleman writes in eqn 7.57 that

$$|\psi(-\infty)\rangle^{\text{in}} = \lim_{t'\to-\infty} e^{iH_0t'}e^{-iHt'}|\psi\rangle \tag{8}$$

$$|\phi(\infty)\rangle^{\text{out}} = \lim_{t''\to\infty} e^{iH_0t''}e^{-iHt''}|\phi\rangle \tag{9}$$

Here, he seems to have changed to the interaction picture, although the earlier discussion used the Schrödinger picture. In the interaction picture, Coleman says these equations are equivalent to

$$|\psi(-\infty)\rangle_I^{\text{in}} = \lim_{t'\to-\infty} U_I\left(0,t'\right)|\psi\rangle \tag{10}$$

$$|\phi(\infty)\rangle^{\text{out}} = \lim_{t''\to-\infty} U_I\left(0,t''\right)|\phi\rangle \tag{11}$$

However, the interaction evolution operator is given by (Coleman's eqn 7.31; note that the 0 and t have swapped places):

$$U_I(t,0) = e^{iH_0t}e^{-iHt} \tag{12}$$

so it appears that the U_I operator in 10 is actually $U^{-1}(0,t')$. I'm really not sure what is going on here, so any comments would be welcome.

Coleman concludes by saying that the S-matrix operator is then

$$\mathrm{S} = \lim_{\substack{T\to\infty\\ \Delta\to\infty\\ (\Delta/T)\to0}} U_I(\infty,-\infty) \tag{13}$$

which follows provided 10 is, in fact, correct, as shown in eqn 7.59. That is, we take the limit in which the interaction is always on ($T\to\infty$), even though it takes an infinite time to become switched on ($\Delta\to\infty$), but the time in which it takes the interaction to become switched on becomes negligible relative to the time interaction is actually on ($\Delta/T\to0$).

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