WICK DIAGRAMS

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Reference: Sidney Coleman, *Quantum Field Theory: Lectures of Sidney Coleman* (edited by Bryan Gin-ge Chen *et al.*), World Scientific, 2019. Section 8.3.

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We'll begin with a quick review of what we're trying to do. We're trying to find the evolution operator $U_I(\infty,-\infty)$ in the interaction picture. This is given by Dyson's formula using a time-ordered exponential.

$$U_I\left(t,t'\right)=T\exp\left(-i\int_{t'}^{t}dt''H_I\left(t''\right)\right) \tag{1}$$

In practice, we usually need to expand the exponential as a series and evaluate each term in the series up to some cutoff term, using perturbation theory. The nth term in the expansion has the form

$$\frac{(-i)^n}{n!}T\int_1\int_2\ldots\int_n dt_1dt_2\ldots dt_nH_{I1}H_{I2}\ldots H_{In} \tag{2}$$

where $H_{Ii}=H_I(t_i)$.

We've seen that we can use Wick's theorem to express a time-ordered product of free fields in terms of a sum of normal ordered products with contractions. In the interaction picture, field operators are free fields since they are obtained by considering only the free part of the hamiltonian. For an operator A_S in the Schrödinger picture, we form its interaction picture version according to

$$A_I(t)=e^{iH_0(p_S,q_S)t}A_S(t)e^{-iH_0(p_S,q_S)t} \tag{3}$$

Coleman uses as his example the Lagrangian

$$\mathscr{L}=\frac{1}{2}(\partial^\mu\phi)(\partial_\mu\phi)-\frac{1}{2}\mu^2\phi^2+\partial^\mu\psi^*\partial_\mu\psi-m^2\psi^*\psi-g\phi\psi^*\psi \tag{4}$$

The first 4 terms are the terms for a free Lagrangian where the ϕ field represents mesons and the ψ and ψ^* fields represent nucleons and antinucleons. The interaction is contained in the last term, which in the Hamiltonian takes the form

$$\mathscr{H}_I = gf(t)\,\psi^*\psi\phi \tag{5}$$

where $f(t)$ is a damping function which turns the interaction on and off at some times in the past and future. $\mathscr{H}_I$ is a hamiltonian density, so to use it in 2 we need to integrate it over 3-d space, so we have for the nth term in the expansion

$$\frac{(-i)^n}{n!}T\int_1\int_2\ldots\int_n d^4x_1d^4x_2\ldots d^4x_n\mathscr{H}_{I1}\mathscr{H}_{I2}\ldots\mathscr{H}_{In} \tag{6}$$

Using the example 5, the second order term is Coleman's eqn 8.35:

$$\frac{(-ig)^2}{2!}\int d^4x_1d^4x_2T\left(\psi_1^*\psi_1\phi_1\psi_2^*\psi_2\phi_2\right) \tag{7}$$

where the subscript on a field operator indicates which of x_1 and x_2 it depends on. At this point, we convert the time ordered product in the integrand into a sum of normal ordered products and contractions using Wick's theorem. It is at this stage that Coleman introduces Wick diagrams as a diagrammatic way of viewing each term in this sum.

The second order term 7 contains 6 fields, so the Wick expansion will contain terms with no contractions and 6 active fields, one contraction and 4 active fields, 2 contractions and 2 active fields, and 3 contractions with no active fields.

If we contract the two ϕ fields, then we're left with the 4 active nucleon-antinucleon fields in the normal ordered product

$$:\psi_1^*\psi_1\overbrace{\phi_1\psi_2^*\psi_2\phi_2}: \tag{8}$$

This term thus gives a term in the expansion of the integral 7

$$\frac{(-ig)^2}{2!}\int d^4x_1d^4x_2f(t_1)\,:\psi_1^*\psi_1\overbrace{\phi_1\psi_2^*\psi_2\phi_2}: \tag{9}$$

The original definition of these fields is (using a † instead of a *):

$$\psi(x) = \int\frac{d^3\boldsymbol{p}}{\sqrt{(2\pi)^3}\sqrt{2\omega_{\boldsymbol{p}}}}\left(b_{\boldsymbol{p}}e^{-ip\cdot x}+c_{\boldsymbol{p}}^\dagger e^{ip\cdot x}\right) \tag{10}$$

$$\psi^\dagger(x) = \int\frac{d^3\boldsymbol{p}}{\sqrt{(2\pi)^3}\sqrt{2\omega_{\boldsymbol{p}}}}\left(b_{\boldsymbol{p}}^\dagger e^{ip\cdot x}+c_{\boldsymbol{p}}e^{-ip\cdot x}\right) \tag{11}$$

The ψ field contains b operators which annihilate a nucleon, and $c^\dagger$ operators which create antinucleons. Conversely, the $\psi^\dagger$ field contains $b^\dagger$ for creating nucleons and c for annihilating antinucleons. Thus if we start and

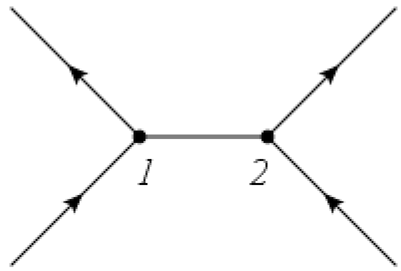

FIGURE 1. The diagram corresponding to $:\psi_1^*\psi_1\phi_1\psi_2^*\psi_2\phi_2:$

end with a 2-nucleon (or antinucleon) state, the matrix element of these two states with a product of 4 normal ordered nucleon fields will be nonzero. Coleman shows that the possible interactions are

$$N+N \to N+N \tag{12}$$

$$N+\bar{N} \to N+\bar{N} \tag{13}$$

$$\bar{N}+\bar{N} \to \bar{N}+\bar{N} \tag{14}$$

In the first reaction, for example, the two ψ fields in 8 annihilate the two incoming nucleons, one at x_1 and the other at x_2. The two ψ^* operators then create the two outgoing nucleons, again at x_1 and x_2. In the second reaction, one of the ψ fields annihilates (at x_1, say) the incoming nucleon and one of the ψ^* fields annihilates the incoming antinucleon at x_2. The other ψ field creates the outgoing antinucleon at x_2 (since the ψ_1 field was used to annihilate the incoming nucleon) and the other ψ^* field creates the outgoing nucleon at x_1.

The reaction $N+N \to \bar{N}+\bar{N}$ is not possible, since it is only the ψ field that can annihilate incoming nucleons and create outgoing antinucleons, so we'd need 4 ψ fields (and no ψ^* fields) to make this reaction work. This reaction would violate charge conservation if it were possible.

A Wick diagram for this example draws a vertex for each x_i location that appears as an integration variable in 7. For an active ψ field, we draw an edge with an arrow pointing towards the vertex, and for each ψ^* field we draw an edge with an arrow pointing away from its vertex. An active ϕ field is an edge without an arrow leading away from its vertex. A contraction is indicated by connecting the two edges for the contracted fields. Thus for the term 8, the diagram is in Fig. 1.

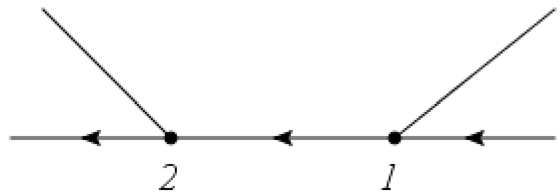

FIGURE 2. The diagram corresponding to $:\overbracket{\psi_1^*\psi_1\phi_1\psi_2^*\psi_2}\phi_2:$

The two incoming edges at the bottom represent ψ_1 and ψ_2, the two outgoing edges at the top represent ψ_1^* and ψ_2^* and the horizontal line represents the contraction $\overbracket{\phi_1\phi_2}$.

Coleman's second example of a Wick diagram is for the case where ψ_1^* is contracted with ψ_2, so the integral is

$$\frac{(-ig)^2}{2!}\int d^4x_1 d^4x_2 f(t_1) :\overbracket{\psi_1^*\psi_1\phi_1\psi_2^*\psi_2}\phi_2: \tag{15}$$

The corresponding Wick diagram is in Fig. 2. [I think Fig. 8.4 in the book has the vertex labels the wrong way round, since the outgoing edge (ψ_1^*) from vertex 1 is contracted with the incoming edge (ψ_2) of vertex 2.]

The single edge connecting vertices 1 and 2 represents the contraction $\overbracket{\psi_1^*\psi_2}$.

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