EXACT SOLUTION OF MODEL 1

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Reference: Sidney Coleman, *Quantum Field Theory: Lectures of Sidney Coleman* (edited by Bryan Gin-ge Chen *et al.*), World Scientific, 2019. Section 8.5.

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As a first example of the theorem relating the evolution operator to connected Wick diagrams, which is

$$U_I(\infty,-\infty) = \sum(\text{all Wick diagrams}) =:\exp\left(\sum \text{connected Wick diagrams}\right): \tag{1}$$

Coleman considers the simple interaction hamiltonian density

$$\mathscr{H}_I = g\rho(x)\phi(x) \tag{2}$$

where g is a numerical parameter giving the strength of the interaction, $\rho(x)$ is a numerical function (not an operator) which goes to zero in all four dimensions of spacetime and $\phi(x)$ is the usual field operator.

The useful feature of this system is that there are only 2 connected diagrams, no matter what order in perturbation theory we consider. The simplest diagram is a single vertex with a single edge and the other is two vertices connected by a contraction, as in Fig. 1.

The symmetry numbers are $S(D_1) = 1$ (since only one diagram is possible with a single vertex and a single edge) and $S(D_2) = 2$ (since interchanging the vertices 1 and 2 produces the same diagram). All higher order diagrams can be split into collections of these two connected diagrams.

Diagram 2 has the corresponding integral, which we'll call O_2:

$$O_2 = (-ig)^2 \int d^4x_1 d^4x_2 \overline{\phi(x_1)\phi}(x_2)\rho(x_1)\rho(x_2) \tag{3}$$

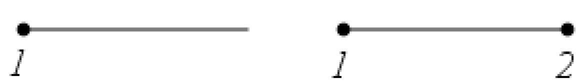

FIGURE 1. The two connected diagrams for Model 1.

The contraction is just a number, so O_2 is the integral of purely numerical functions (no operators) and thus is itself some complex number, which Coleman writes as

$$O_2 = -\alpha + i\beta \tag{4}$$

The operator O_1 corresponding to diagram D_1, however, does contain an uncontracted operator. Coleman shows in eqns 8.63 to 8.67 that this comes out to

$$O_1 = \int d^3\boldsymbol{p}\left[-h(\boldsymbol{p})^* a_{\boldsymbol{p}} + h(\boldsymbol{p})\, a_{\boldsymbol{p}}^\dagger\right] \tag{5}$$

where $a_{\boldsymbol{p}}$ and $a_{\boldsymbol{p}}^\dagger$ are the usual annihilation and creation operators for the scalar meson field, and

$$h(\boldsymbol{p}) = \frac{-ig\tilde{\rho}(\boldsymbol{p},\omega_{\boldsymbol{p}})}{(2\pi)^{3/2}\sqrt{2\omega_{\boldsymbol{p}}}} \tag{6}$$

with $\tilde{\rho}(p)$ being the Fourier transform of $\rho(x)$. Since we have only one meson in diagram D_1, the four-momentum p must be 'on the mass shell', that is, it must satisfy

$$p^2 = \mu^2 \tag{7}$$

$$\omega_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2 + \mu^2} \tag{8}$$

If the Fourier transform $\tilde{\rho}$ vanishes on the mass shell in 6, then $h = 0$ and therefore $O_1 = 0$. This is conservation of energy in this single-meson model. Because there is only one connected diagram in which any mesons are produced, and that diagram D_1 produces only a single meson, the only processes in which mesons are produced produces them one at a time. If $\mathscr{H}_i$ contained higher-order terms such as ϕ^2, then we could have two meson lines emanating from a single vertex and thus could produce 2 mesons in a single interaction.

Coleman now sets about evaluating O_1. The discussion is fairly easy to follow, but there are a few points that need some clarification.

Coleman considers the system starting from the ground state (vacuum state) $|0\rangle$ and shows in eqn 8.68 that the state $|\psi\rangle$ obtained by operating on this ground state with $S = U_I(\infty,-\infty)$ is

$$|\psi\rangle = A\exp\left[\int d^3\boldsymbol{p}h(\boldsymbol{p})\, a_{\boldsymbol{p}}^\dagger\right]|0\rangle \tag{9}$$

where A is a normalization constant.

We now write $|\psi\rangle$ as an integral over the basis states $|\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_n\rangle$ where

$$|\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_n\rangle = a^\dagger_{\boldsymbol{p}_1} a^\dagger_{\boldsymbol{p}_2} \ldots a^\dagger_{\boldsymbol{p}_n} |0\rangle \tag{10}$$

A general state $|\psi\rangle$ can consist of any number of mesons even though we can produce only a single meson at a time because $|\psi\rangle$ is the result of applying *all* Wick diagrams (including the disconnected ones) to the ground state.

The expansion in terms of this basis is eqn 8.70

$$|\psi\rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^3\boldsymbol{p}_1 d^3\boldsymbol{p}_2 \ldots d^3\boldsymbol{p}_n \psi^{(n)}(\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_n) |\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_n\rangle \tag{11}$$

where

$$\psi^{(n)}(\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_n) \equiv \langle \boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_n | \psi \rangle = \langle \boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_n | S | 0 \rangle \tag{12}$$

In the arguments that follow, it's important to keep track of the presence or absence of the $n!$ factor, which can be a bit confusing. In 11, the multiple integral over the n different momenta would overcount the contribution from the nth order term, since if we interchange any pair of momenta, we get the same result. Since there are $n!$ ways of ordering the n momenta, we include the $\frac{1}{n!}$ to ensure that we get only one contribution from the nth order term.

Coleman now compares 9 with 11 to find the relation between $\psi^{(n)}$ and h. We get from 9

$$|\psi\rangle = A \sum_{n=0}^{\infty} \frac{1}{n!} \left[\int d^3\boldsymbol{p} h(\boldsymbol{p}) a^\dagger_{\boldsymbol{p}} \right]^n \tag{13}$$

Doing the comparison gives

$$\psi^{(0)} = A \tag{14}$$

$$\psi^{(1)} = A h(\boldsymbol{p}) \tag{15}$$

$$\psi^{(2)} = A h(\boldsymbol{p}_1) h(\boldsymbol{p}_2) \tag{16}$$

How do we get the $\psi^{(2)}$ equation? From 9, the second order term is

$$A \frac{1}{2!} \int d^3\boldsymbol{p}_1 d^3\boldsymbol{p}_2 h(\boldsymbol{p}_1) h(\boldsymbol{p}_1) a^\dagger_{\boldsymbol{p}_1} a^\dagger_{\boldsymbol{p}_2} |0\rangle = A \frac{1}{2!} \int d^3\boldsymbol{p}_1 d^3\boldsymbol{p}_2 h(\boldsymbol{p}_1) h(\boldsymbol{p}_1) |\boldsymbol{p}_1, \boldsymbol{p}_2\rangle \tag{17}$$

Comparing this with 11 we see that the 2! has been separated out in both expressions, so we get 16.

The probability $P(n)$ of finding n mesons in the final state is the square modulus of the nth order term in 11. Coleman gives this as

$$P(n)=\frac{1}{n!}\int d^{3}\boldsymbol{p}_{1}d^{3}\boldsymbol{p}_{2}\ldots d^{3}\boldsymbol{p}_{n}\left|\psi^{(n)}(\boldsymbol{p}_{1},\boldsymbol{p}_{2},\ldots,\boldsymbol{p}_{n})\right|^{2} \tag{18}$$

Again, if we left out the $\frac{1}{n!}$, we would be overcounting the number of final states, since interchanging any two momenta gives the same result after integration.

Coleman concludes by working out a couple of parameters. The normalization constant A comes out to

$$|A|^{2}=\exp\left(-\int d^{3}\boldsymbol{p}\,|h(\boldsymbol{p})|^{2}\right) \tag{19}$$

which makes α from 4 come out to

$$\alpha=\int d^{3}\boldsymbol{p}\,|h(\boldsymbol{p})|^{2} \tag{20}$$

and thus the probability $P(n)$ comes out to be a Poisson distribution

$$P(n)=e^{-\alpha}\frac{\alpha^{n}}{n!} \tag{21}$$

[Details in eqns 8.74 to 8.78.]

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