PHASE VELOCITY AND GROUP VELOCITY IN A WAVE PACKET

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A monochromatic wave has velocity called the *phase velocity* given by

$$v_p = \frac{\omega}{k} \tag{1}$$

where ω is the frequency and $k = 2\pi/\lambda$ is the wave number. However, if we have a compound wave that is composed of individual waves with a range of frequencies, each individual wave has a velocity given by 1, but the amplitudes of the waves add up to produce a wave packet which has a velocity all its own. This velocity is called the *group velocity* and is usually different from the individual phase velocities of the waves that make up the packet.

This effect arises from the fact that typically the frequency is a function of the wave number: $\omega = \omega(k)$. Suppose we have a wave packet made up of a range of individual waves. We can write this as

$$\psi(x,t) = \int A(k)\, e^{i(kx-\omega t)} dk \tag{2}$$

where $A(k)$ is a function giving the contribution to the packet of waves with wave number k. Now suppose that most of these waves have values of k that are close to some value k_0. In that case, we can expand $\omega(k)$ and keep only the first order term:

$$\omega(k_0 + \Delta k) = \omega_0 + \Delta k \omega_0' \tag{3}$$

where

$$\omega_0' \equiv \left.\frac{d\omega}{dk}\right|_{k_0} \tag{4}$$

Plugging this into 2 we get

$$\psi(x,t) = e^{i(k_0 x - \omega_0 t)} \int A(k)\, e^{i\left(\Delta k\cdot x - \Delta k\cdot \omega_0' t\right)} d(\Delta k) \tag{5}$$

We can see that the wave packet is composed of a monochromatic wave represented by the exponential outside the integral modulated by the integral factor. The speed of the monochromatic wave is just the phase velocity of the main wave:

$$v_p = \frac{\omega_0}{k_0} \tag{6}$$

The velocity of the modulation is now a constant given by

$$v_g = \frac{\Delta k \cdot \omega_0'}{\Delta k} = \left.\frac{d\omega}{dk}\right|_{k_0} \tag{7}$$

It's this latter velocity that is the group velocity. This derivation relies on the waves making up the packet all having wave numbers (and hence wavelengths) lying close to each other.

Example 1. A wave travelling on the surface of water has a phase velocity that is proportional to the square root of its wavelength (provided λ is less than the depth of the water). That is,

$$v_p = \frac{\omega}{k} = A\sqrt{\lambda} \tag{8}$$

$$= A\sqrt{\frac{2\pi}{k}} \tag{9}$$

$$\omega(k) = A\sqrt{2\pi k} \tag{10}$$

$$v_g = \frac{d\omega}{dk} \tag{11}$$

$$= \frac{1}{2}A\sqrt{\frac{2\pi}{k}} \tag{12}$$

$$= \frac{1}{2}v_p \tag{13}$$

Thus the phase velocity of deep water waves is twice the group velocity.

Example 2. We've seen that a free particle can be represented in quantum mechanics by a superposition of waves, each of which has the form

$$\Psi(x,t) = Ae^{i(px-Et)/\hbar} \tag{14}$$

where p is the momentum and E is the energy, given by

$$E = \frac{p^2}{2m} \tag{15}$$

In terms of frequency and wave number, we have

$$k = \frac{p}{\hbar} \tag{16}$$

$$\omega = \frac{p^2}{2\hbar m} \tag{17}$$

$$= \frac{\hbar k^2}{2m} \tag{18}$$

$$v_g = \frac{d\omega}{dk} = \frac{\hbar k}{m} = \frac{p}{m} \tag{19}$$

$$v_p = \frac{\omega}{k} = \frac{p}{2m} = \frac{v_g}{2} \tag{20}$$

So in this case, the phase velocity is half the group velocity. Classically a particle's velocity is given by $v = p/m$ so it is the quantum group velocity that corresponds to classical velocity.