

RECTANGULAR WAVE GUIDES - TRANSVERSE ELECTRIC WAVES

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The wave equations for an electromagnetic wave in a wave guide are

$$[\partial_{xx} + \partial_{yy} + \omega^2/c^2 - k^2] E_z = 0 \quad (1)$$

$$[\partial_{xx} + \partial_{yy} + \omega^2/c^2 - k^2] B_z = 0 \quad (2)$$

If $E_z = 0$ we have a *transverse electric* or TE wave, so we need to solve only the B_z equation. For a rectangular wave guide with dimensions of a in the x direction and b in the y direction, we can use separation of variables to get a solution (the technique is mathematically the same as that used in solving the infinite square well in quantum mechanics). That is, we assume that

$$B_z(x, y) = X(x)Y(y) \quad (3)$$

and substitute this into 2, then divide through by XY :

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{\omega^2}{c^2} - k^2 = 0 \quad (4)$$

Because this equation must hold for all values of x and y , the two derivative terms must separately be constant, so we have

$$X'' = -k_x^2 X \quad (5)$$

$$Y'' = -k_y^2 Y \quad (6)$$

for some constants k_x and k_y , which satisfy

$$-k_x^2 - k_y^2 + \frac{\omega^2}{c^2} - k^2 = 0 \quad (7)$$

The general solution to these ODEs can be written as either trigonometric functions or complex exponentials. If we choose trig functions, we have

$$X(x) = A \sin k_x x + B \cos k_x x \quad (8)$$

$$Y(y) = C \sin k_y y + D \cos k_y y \quad (9)$$

The boundary conditions require that at $x = 0$ and $x = a$

$$B_1^\perp - B_2^\perp = 0 \quad (10)$$

The component of \mathbf{B} perpendicular to the walls of the guide in the x direction is B_x , and since $\mathbf{B} = 0$ inside the conducting wall, this condition requires $B_x = 0$. In our derivation of the wave equation for a wave guide, we found that

$$B_x = \frac{i}{\omega^2/c^2 - k^2} \left(k \partial_x B_z - \frac{\omega}{c^2} \partial_y E_z \right) \quad (11)$$

and since $E_z = 0$, we must also have $\partial_x B_z = 0$, which in turn requires that

$$X'(0) = X'(a) = 0 \quad (12)$$

From 8, this means that $A = 0$ and

$$-k_x B \sin k_x a = 0 \quad (13)$$

so

$$k_x = \frac{m\pi}{a} \quad (14)$$

for m equal to a non-negative integer.

Exactly the same analysis on Y gives us

$$k_y = \frac{n\pi}{b} \quad (15)$$

so the separation of variables solution gives us

$$B_z(x, y) = B_0 \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \quad (16)$$

This is known as the TE_{mn} mode. From 7 we get the wave number

$$k = \sqrt{\frac{\omega^2}{c^2} - \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)} \quad (17)$$

In fact, at least one of m or n must be non-zero, as we can see by the following argument. Suppose $m = n = 0$. Then $k = \frac{\omega}{c}$. We need the results we got in the previous post:

$$\partial_y E_z - ikE_y = i\omega B_x \quad (18)$$

$$-\partial_x E_z + ikE_x = i\omega B_y \quad (19)$$

$$\partial_x E_y - \partial_y E_x = i\omega B_z \quad (20)$$

$$\partial_y B_z - ikB_y = -i\frac{\omega}{c^2} E_x \quad (21)$$

$$-\partial_x B_z + ikB_x = -i\frac{\omega}{c^2} E_y \quad (22)$$

$$\partial_x B_y - \partial_y B_x = -i\frac{\omega}{c^2} E_z \quad (23)$$

With $k = \frac{\omega}{c}$ and $E_z = 0$ we get from 21 and 19

$$\partial_y B_z - i\frac{\omega}{c} B_y = -i\frac{\omega}{c^2} E_x \quad (24)$$

$$i\frac{\omega}{c^2} E_x = i\frac{\omega}{c} B_y \quad (25)$$

Adding these equations gives us

$$\partial_y B_z = 0 \quad (26)$$

Similarly, from 22 and 18 we get

$$\partial_x B_z = 0 \quad (27)$$

Therefore, B_z is constant in both the x and y directions, so it is a constant overall. To find what constant this is, we can use Faraday's law in integral form:

$$\oint \mathbf{E} \cdot d\boldsymbol{\ell} = -\frac{d\Phi}{dt} = -\int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{a} \quad (28)$$

The area of integration on the RHS that we'll choose is a cross-section of the wave guide, so the path of integration on the LHS is around the rectangular boundary of the guide. We know that

$$\mathbf{B} = \mathbf{B}_0 e^{i(kz - \omega t)} \quad (29)$$

so

$$\frac{\partial \mathbf{B}}{\partial t} = -i\omega \mathbf{B}_0 e^{i(kz - \omega t)} \quad (30)$$

In this case $d\mathbf{a}$ points in the z direction, so we get

$$\oint \mathbf{E} \cdot d\boldsymbol{\ell} = i\omega e^{i(kz - \omega t)} \int B_z da \quad (31)$$

$$= i\omega ab e^{i(kz - \omega t)} B_z \quad (32)$$

since B_z is constant over the area of integration.

Since we've chosen the path of integration to be the boundary of the wave guide, we can use the boundary condition

$$\mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} = 0 \quad (33)$$

The field inside the conducting wall of the wave guide is zero, so the parallel field at the boundary must also be zero and, since $\mathbf{E} \cdot d\boldsymbol{\ell} = E^{\parallel} d\ell$, the line integral must also be zero, so we must have

$$B_z = 0 \quad (34)$$

That is, if $m = n = 0$, both the electric and magnetic fields must be transverse (known as TEM mode). Griffiths shows in section 9.5 of his book that in this case, for a hollow wave guide of any cross section, the electric field must actually be zero everywhere, which means no wave can propagate down the guide. Thus the TE_{00} mode cannot exist in a hollow wave guide.