POISSON BRACKETS AND HAMILTON’S EQUATIONS OF MOTION

Although I’ve looked at Poisson brackets before, it’s worth going through G & R’s treatment as it is a fair bit simpler and gives clearer results.

First, we need the time derivative of a functional. In the simplest case, a functional $F[\phi]$ depends on a function $\phi$ which in turn depends on an independent variable $x$. The functional itself does not depend on $x$, however, usually because $F$ is defined as the integral of $\phi(x)$ over some range of $x$ values, so the dependence on $x$ disappears in the integration.

We can generalize things a bit by taking $\phi$ as a function of two variables, say $x$ and $t$. If $F$ is defined in the same way (say, as an integral of $\phi$ over $x$), then the variable $t$ also appears in the functional, so we can write this as $F(t)$, which is

$$F(t) = \int dx \, g(\phi(x,t))$$  \hspace{1cm} (1)

where $g(\phi)$ is some function of $\phi$. Since $F$ now depends on $t$, we can take the derivative $dF/dt$ which comes out to

$$\dot{F} \equiv \frac{dF}{dt} = \int dx \, \frac{dg}{d\phi} \frac{\partial \phi}{\partial t} = \int dx \, \frac{dg}{d\phi} \dot{\phi}(x,t)$$  \hspace{1cm} (2)

As we’ve seen before, the functional derivative of $F$ in this case is

$$\frac{\delta F(t)}{\delta \phi(y,t)} = \frac{dg(\phi(y,t))}{d\phi}$$  \hspace{1cm} (3)

where the notation means that we evaluate the derivative on the RHS at the point $(y,t)$. Using this result, we can therefore write $\dot{F}$ as

$$\dot{F}(t) = \int dx \, \frac{\delta F(t)}{\delta \phi(x,t)} \dot{\phi}(x,t)$$  \hspace{1cm} (4)

We can generalize this to 4-d space time, so that $x$ now indicates the four-vector $x = (x,t)$, and the integral is over 3-d space:
\[
\dot{F}(t) = \int d^3x \frac{\delta F(t)}{\delta \phi(x)} \dot{\phi}(x)
\]  
(5)

Generalizing even further, we can make \( F \) a functional of two fields, \( \phi \) and \( \pi \), so we get

\[
\dot{F}(t) = \int d^3x \left[ \frac{\delta F(t)}{\delta \phi(x)} \dot{\phi}(x) + \frac{\delta F(t)}{\delta \pi(x)} \dot{\pi}(x) \right]
\]  
(6)

Interpreting \( \phi \) as the field and \( \pi \) as its conjugate momentum, we can now use Hamilton’s equations of motion

\[
\dot{\phi} = \frac{\delta H}{\delta \pi}
\]  
(7)

\[
\dot{\pi} = -\frac{\delta H}{\delta \phi}
\]  
(8)

We get

\[
\dot{F}(t) = \int d^3x \left[ \frac{\delta F(t)}{\delta \phi(x)} \frac{\delta H}{\delta \pi} - \frac{\delta F(t)}{\delta \pi(x)} \frac{\delta H}{\delta \phi} \right]
\]  
(9)

The quantity on the RHS is defined to be the Poisson bracket:

\[
\{ F, H \}_{PB} \equiv \int d^3x \left[ \frac{\delta F(t)}{\delta \phi(x)} \frac{\delta H}{\delta \pi} - \frac{\delta F(t)}{\delta \pi(x)} \frac{\delta H}{\delta \phi} \right]
\]  
(10)

We thus have the general result that the time derivative of a functional is equal to its Poisson bracket with the Hamiltonian:

\[
\dot{F}(t) = \{ F, H \}_{PB}
\]  
(11)

We can use this result in a rather curious way to re-derive Hamilton’s equations of motion. We first observe that we can write the field \( \phi \) as an integral:

\[
\phi(x, t) = \int d^3x' \phi(x', t) \delta^3(x - x')
\]  
(12)

This effectively defines an ordinary function \( \phi \) as a functional depending on itself. In this case, both \( x \) and \( t \) are parameters that are present on both sides of the equation; it is the dummy variable \( x' \) that is the variable of integration.

Taking the variation on both sides, we get

\[
\delta \phi(x, t) = \int d^3x' \delta \phi(x', t) \delta^3(x - x')
\]  
(13)
[Be careful not to get the $\delta$s confused here: $\delta \phi$ is a variation of the function $\phi$ while $\delta^3$ is the 3-d delta function.] Comparing this to the definition of the functional derivative

$$\delta F[\phi] \equiv \int d^3x \frac{\delta F[\phi]}{\delta \phi(x)} \delta \phi(x)$$

we see that we have the functional derivative of $\phi$ with respect to itself:

$$\frac{\delta \phi(x,t)}{\delta \phi(x',t)} = \delta^3(x-x')$$

(15)

We could use the same argument on the conjugate momentum, so we also have

$$\frac{\delta \pi(x,t)}{\delta \pi(x',t)} = \delta^3(x-x')$$

(16)

Since $\phi$ and $\pi$ are independent fields

$$\frac{\delta \pi(x,t)}{\delta \phi(x',t)} = \frac{\delta \phi(x,t)}{\delta \pi(x',t)} = 0$$

(17)

We can now use [11] to find the time derivatives of $\phi$ and $\pi$ by treating them as functionals:

$$\dot{\phi}(x,t) = \{\phi(x,t), H\}$$

$$= \int d^3x' \left[ \frac{\delta \phi(x,t)}{\delta \phi(x',t)} \frac{\delta H(t)}{\delta \pi(x',t)} - \frac{\delta \phi(x,t)}{\delta \pi(x',t)} \frac{\delta H(t)}{\delta \phi(x',t)} \right]$$

$$= \int d^3x' \left[ \delta^3(x-x') \frac{\delta H(t)}{\delta \pi(x',t)} - 0 \right]$$

$$= \frac{\delta H(t)}{\delta \pi(x,t)}$$

(18) (19) (20) (21)

This gives the first Hamilton equation of motion [7]. We can work out the second equation similarly:
\[ \dot{\pi}(x,t) = \{\pi(x,t), H\} \]

(22)

\[ = \int d^3x' \left[ \frac{\delta \pi(x,t)}{\delta \phi(x',t)} \frac{\delta H(t)}{\delta \pi(x',t)} - \frac{\delta \pi(x,t)}{\delta \phi(x',t)} \frac{\delta H(t)}{\delta \pi(x',t)} \right] \]

(23)

\[ = \int d^3x' \left[ 0 - \delta^3(x-x') \frac{\delta H(t)}{\delta \phi(x',t)} \right] \]

(24)

\[ = -\frac{\delta H(t)}{\delta \phi(x,t)} \]

(25)

Finally, we can work out the Poisson brackets of the fields with each other, using the definition 10 and the results above.

\[ \{\phi(x,t), \pi(x',t)\}_{PB} \equiv \int d^3x'' \left[ \frac{\delta \phi(x,t)}{\delta \phi(x'',t)} \frac{\delta \pi(x',t)}{\delta \pi(x'',t)} - \frac{\delta \phi(x,t)}{\delta \pi(x',t)} \frac{\delta \pi(x'',t)}{\delta \phi(x'',t)} \right] \]

(26)

\[ = \int d^3x'' \left[ \delta^3(x-x'') \delta^3(x'-x'') - 0 \right] \]

(27)

\[ = \delta^3(x-x') \]

(28)

The other two Poisson brackets are zero because of 17:

\[ \{\phi(x,t), \phi(x',t)\}_{PB} \equiv \int d^3x'' \left[ \frac{\delta \phi(x,t)}{\delta \phi(x'',t)} \frac{\delta \phi(x',t)}{\delta \phi(x'',t)} - \frac{\delta \phi(x,t)}{\delta \phi(x',t)} \frac{\delta \phi(x'',t)}{\delta \phi(x'',t)} \right] \]

(29)

\[ = \int d^3x'' [0 - 0] \]

(30)

\[ = 0 \]

(31)

\[ \{\pi(x,t), \pi(x',t)\}_{PB} \equiv \int d^3x'' \left[ \frac{\delta \pi(x,t)}{\delta \pi(x'',t)} \frac{\delta \pi(x',t)}{\delta \pi(x'',t)} - \frac{\delta \pi(x,t)}{\delta \pi(x',t)} \frac{\delta \pi(x'',t)}{\delta \pi(x'',t)} \right] \]

(32)

\[ = \int d^3x'' [0 - 0] \]

(33)

\[ = 0 \]

(34)

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