The Klein-Gordon equation is a relativistic wave equation which looks like this

\[
\left( \frac{\partial^2}{c^2 \partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} + \frac{m_0 c^2}{\hbar^2} \right) \psi = 0 \tag{1}
\]

Plane wave solutions of this equation are

\[
\psi = \exp \left( \frac{i}{\hbar} (p \cdot x - Et) \right) \tag{2}
\]

We would like to show that in the nonrelativistic limit, this equation reduces to the Schrödinger equation. To do this, we write the wave function \(\psi\) as the product of two factors:

\[
\psi(r,t) = \phi(r,t) e^{-im_0 c^2 t/\hbar} \tag{3}
\]

That is, we have split the portion of the energy due to the rest energy of the particle off into its own factor, so that the energy contained in the \(\phi\) factor is the kinetic energy \(E' = E - m_0 c^2\). For a nonrelativistic particle, we would expect \(E' \ll m_0 c^2\). If we operate on \(\phi\) with the energy operator \(\hat{E} = i\hbar \frac{\partial}{\partial t}\), we would then expect that this would give

\[
\left| i\hbar \frac{\partial \phi}{\partial t} \right| \approx E' \phi \ll m_0 c^2 \phi \tag{4}
\]

Taking the time derivative of \(\phi\), we therefore have, using \(\frac{\partial^2 \phi}{\partial t^2} \approx \frac{\partial \phi}{\partial t} \approx \frac{im_0 c^2}{\hbar} \phi e^{-im_0 c^2 t/\hbar}

\[
\frac{\partial \psi}{\partial t} = \left( \frac{\partial \phi}{\partial t} - i \frac{m_0 c^2}{\hbar} \phi \right) e^{-im_0 c^2 t/\hbar} \tag{5}
\]

\approx -i \frac{m_0 c^2}{\hbar} \phi e^{-im_0 c^2 t/\hbar} \tag{6}

We now need to take the second derivative of \(\psi\), and at this point, Greiner appears to make the assumption that we can neglect \(\frac{\partial^2 \phi}{\partial t^2}\). It’s not entirely
clear to me how we can justify this assumption, but if we go with it, we get, using the product rule on

$$\frac{\partial^2 \psi}{\partial t^2} = \frac{\partial}{\partial t} \left[ \left( \frac{\partial \phi}{\partial t} - i \frac{m_0 c^2}{\hbar} \phi \right) e^{-im_0 c^2 t/\hbar} \right]$$

(7)

$$\approx -i \frac{m_0 c^2}{\hbar} \frac{\partial \phi}{\partial t} e^{-im_0 c^2 t/\hbar} - i \frac{m_0 c^2}{\hbar} \left( \frac{\partial \phi}{\partial t} - i \frac{m_0 c^2}{\hbar} \phi \right) e^{-im_0 c^2 t/\hbar}$$

(8)

$$= - \left( \frac{2m_0 c^2}{\hbar} \frac{\partial \phi}{\partial t} + \frac{m_0^2 c^4}{\hbar^2} \phi \right) e^{-im_0 c^2 t/\hbar}$$

(9)

Inserting this back into (1) and cancelling terms gives

$$-i \frac{2m_0}{\hbar} \frac{\partial \phi}{\partial t} - \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \phi = 0$$

(10)

Rearranging terms and multiplying through by $\frac{\hbar^2}{2m_0}$ we get

$$i \hbar \frac{\partial \phi}{\partial t} = - \frac{\hbar^2}{2m_0} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \phi$$

(11)

$$= - \frac{\hbar^2}{2m_0} \nabla^2 \phi$$

(12)

This is just the Schrödinger equation for a free particle (and zero spin), where $\phi$ has taken on the role of the wave function.