KLEIN-GORDON EQUATION IN SCHRÖDINGER FORM - LAGRANGIAN, ENERGY-MOMENTUM TENSOR

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Reference: W. Greiner: Relativistic Quantum Mechanics (Wave Equations); 3rd Edition, Springer-Verlag (2000); Section 1.8; Exercise 1.7.
Post date: 10 Dec 2017.

Greiner’s Exercise 1.7 introduces the Lagrangian density and energy-momentum tensor for the Schrödinger representation of the Klein-Gordon equation. (In the text, he says that it’s for the Feshbach-Villars representation, but all the equations refer to the two-component wave function \( \Psi \) rather than the function \( \Phi \) which is used in the Feshbach-Villars form.)

As usual, the Lagrangian density is just stated (one of these days I’d like to find out if it is possible to actually derive a Lagrangian rather than conjuring them out of thin air) to be

\[
\mathcal{L} = \frac{i\hbar}{\bar{\Psi}} \frac{\partial}{\partial t} \Psi - \frac{\hbar^2}{2m_0} \nabla (\tau_3 + i\tau_2) \nabla \Psi - m_0 c^2 \bar{\Psi} \tau_3 \Psi
\]  

(1)

where the \( \tau_i \) matrices are essentially the Pauli matrices

\[
\tau_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad \tau_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}; \quad \tau_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}; \quad 1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

(2)

and

\[
\bar{\Psi} \equiv \Psi^\dagger \tau_3
\]

(3)

By using the usual technique of varying the action integral, Greiner shows that the Euler-Lagrange equations yield the Schrödinger representation of the Klein-Gordon equation. That is, variation of \( \Psi \) yields the equation

\[
\frac{\partial}{\partial v} \frac{\partial \mathcal{L}}{\partial (\partial_v \Psi_\alpha)} - \frac{\partial \mathcal{L}}{\partial \Psi_\alpha} = 0
\]

(4)

which gives

\[
i\hbar \partial_t \Psi_\alpha = -\frac{\hbar^2}{2m_0} (\tau_3 + i\tau_2) \nabla^2 \Psi_\alpha + m_0 c^2 \tau_3 \Psi_\alpha
\]

(5)
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where $\alpha = 1, 2$ refers to one of the two components of $\Psi$. Variation of $\Psi$ gives the conjugate equation

$$-i\hbar \partial_t \bar{\Psi}_\alpha = -\frac{\hbar^2}{2m_0} (\tau_3 + i\tau_2) \nabla^2 \bar{\Psi}_\alpha + m_0 c^2 \tau_3 \bar{\Psi}_\alpha \tag{6}$$

These two equations are both forms of the Schrödinger representation of the Klein-Gordon equation, which is

$$\left( i\hbar \frac{\partial}{\partial t} - H_f \right) \Psi = 0 \tag{7}$$

with

$$H_f \equiv (\tau_3 + i\tau_2) \frac{\hbar^2 \nabla^2}{2m_0} + \tau_3 m_0 c^2 \tag{8}$$

We’ve seen that an integral of the form

$$\langle L \rangle = \int \Psi^\dagger \tau_3 L \Psi d^3 x \tag{9}$$

is real if the operator $L$ satisfies the generalized hermiticity condition

$$L^H \equiv \tau_3 L^\dagger \tau_3 = L \tag{10}$$

In eqn 1.111, Greiner shows that $H_f$ satisfies this condition. The operator $i\hbar \partial_t$ has the same effect as $H_f$ so it is generally hermitian as well. As Greiner shows in Ex. 1.7, the general hermiticity of $H_f$ ensures that the action integral is real.

The energy-momentum tensor is given by

$$T_{\mu\nu} = -g_{\mu\nu} \mathcal{L} + \sum_\sigma \frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi_\sigma)} \frac{\partial \psi_\sigma}{\partial x^\nu} \tag{11}$$

where the index $\sigma$ ranges over the independent fields. In this case, the fields are $\Psi$ and $\bar{\Psi}$, so we have

$$T_{\mu\nu} = -g_{\mu\nu} \mathcal{L} + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \Psi)} \partial_\nu \Psi + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \bar{\Psi})} \partial_\nu \bar{\Psi} \tag{12}$$

The component $T_{00}$ gives the energy density, and Greiner shows in eqns (4) and (5) that the total energy is given by

Note that the $2 \times 2$ matrices $\tau_3$ and $i\tau_2$ are both real, so their complex conjugates give the same matrix.
\[ E = \int T_{00} d^3x \quad (13) \]
\[ = \int \Psi^\dagger \tau_3 H_f \Psi d^3x \quad (14) \]
\[ = \langle H_f \rangle \quad (15) \]

where the last line follows from the definition of the mean value of an operator in \[\Psi\].

**COMMENTS**

From burning flamer, 22 Oct 2019, 11:55 PM.

You asked if it is possible to derive a Lagrangian. Under some conditions, you can.

If you already have some ODE/PDE, and just want some random (non-unique) Lagrangian to get it, you align the DE to the E-L equations, and integrate. In your case here, start from Equation (5), move the mass term over to the other side. Then you have, on the two sides of the equals sign, the two terms of Equation (4). Integrating (really, you just deduce what works), and you get Equation (1). You might like to impose some symmetry conditions on the resultant Lagrangian.

The above procedure was also how the subject originally came to be. Newtonian mechanics was already invented, and Lagrange wanted to figure out how his new mathematical toolkit could be used to solve problems in Newtonian mechanics. So he worked out what general Lagrangian form would allow him to translate standard Newtonian mechanics problems. That was how the \[L = KE - PE\] prescription originally came about.

You might also want to pay attention to Landau and Lifshitz’s tiny mechanics volume of their textbook series. In the first few pages, they made some symmetry arguments and directly arrived at the Lagrangian, and then derived Newtonian mechanics from there. The 2nd volume also tried to do the same with relativistic mechanics, but they were ugly and nowhere near as nice.

In general, though, you have to postulate something. Either the Lagrangian, or the DE. You "derive" one from the other because they constitute equivalent amount of information. We typically postulate the Lagrangian because the symmetry properties are most obvious, and fruitful, there, and it removes the ambiguity due to the non-uniqueness of the transition from DE to Lagrangian.