KLEIN-GORDON EQUATION - INVARINCE UNDER ELECTROMAGNETIC GAUGE TRANSFORMATIONS

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The Klein-Gordon equation for a particle in an electromagnetic field is given by

\[
\left(p^\mu - \frac{e}{c} A^\mu\right)\left(p_\mu - \frac{e}{c} A_\mu\right) \psi = m_0 c^2 \psi
\]  

where \( A^\mu \) is the electromagnetic four-potential and \( p^\mu = i\hbar \frac{\partial}{\partial x^\mu} \) is the four-momentum. The classical equations of electromagnetism (Maxwell’s equations) are invariant if we impose a gauge transformation on the potential. In four-vector notation, the most general gauge transformation is given by

\[
A'_\mu (x) = A_\mu (x) + \frac{\partial \chi(x)}{\partial x^\mu}
\]  

where \( \chi(x) \) is an arbitrary scalar function of the four-vector \( x^\mu = (ct, \mathbf{x}) \).

What happens if we insert this gauge transformation into the Klein-Gordon equation? We get

\[
\left(p^\nu - \frac{e}{c} A^\nu - \frac{e}{c} \frac{\partial \chi(x)}{\partial x^\nu}\right)\left(p_\mu - \frac{e}{c} A_\mu - \frac{e}{c} \frac{\partial \chi(x)}{\partial x^\mu}\right) \psi = m_0 c^2 \psi
\]  

or, using Greiner’s form:

\[
g^{\mu\nu} \left(p^\nu - \frac{e}{c} A^\nu - \frac{e}{c} \frac{\partial \chi(x)}{\partial x^\nu}\right)\left(p_\mu - \frac{e}{c} A_\mu - \frac{e}{c} \frac{\partial \chi(x)}{\partial x^\mu}\right) \psi = m_0 c^2 \psi
\]  

We can write this transformed equation in the same form as if we define the transformed wave function

\[
\psi' \equiv \psi e^{i \chi / \hbar c}
\]  

We can see this by replacing \( \psi \) by \( \psi' \) in the expression...
\[(p_\mu - \frac{e}{c}A_\mu) \psi \rightarrow (p_\mu - \frac{e}{c}A_\mu) \psi'\]  
\[= \left( \frac{i\hbar}{\mu} - \frac{e}{c}A_\mu \right) \left( \psi e^{i\epsilon \chi / \hbar c} \right)\]  
\[= e^{i\epsilon \chi / \hbar c} \left( \frac{i\hbar}{\mu} \frac{\partial}{\partial x^\mu} - \frac{e}{c}A_\mu \right) \psi + i\hbar \frac{e}{\hbar c} \frac{\partial \chi}{\partial x^\mu} \psi e^{i\epsilon \chi / \hbar c} \]  
\[= e^{i\epsilon \chi / \hbar c} \left( p_\mu - \frac{e}{c}A_\mu - \frac{e}{c} \frac{\partial \chi (x)}{\partial x^\mu} \right) \psi\]  
\[= (p_\mu - \frac{e}{c}A_\mu) \psi = (p_\mu - \frac{e}{c}A_\mu) \psi' = (p_\mu - \frac{e}{c}A_\mu) \left( \psi e^{i\epsilon \chi / \hbar c} \right)\]  
\[= (p_\mu - \frac{e}{c}A_\mu) \psi^\prime = (p_\mu - \frac{e}{c}A_\mu) \left( \psi e^{i\epsilon \chi / \hbar c} \right)\]  
\[= e^{i\epsilon \chi / \hbar c} B'_\mu (\psi) = B_\mu \left( \psi e^{i\epsilon \chi / \hbar c} \right)\]  
\[\text{The parentheses here indicate what the operators } B'_\mu \text{ and } B_\mu \text{ operate on. Now the LHS of (3) can be rewritten as}\]  
\[B'^\mu \left( B'_\mu (\psi) \right)\]  
\[\text{Using (13) we have}\]  
\[e^{i\epsilon \chi / \hbar c} B'^\mu \left( B'_\mu (\psi) \right) = B^\mu \left( B'_\mu (\psi) e^{i\epsilon \chi / \hbar c} \right)\]  
\[\text{However, also from (13)}\]  
\[B'_\mu (\psi) = e^{-i\epsilon \chi / \hbar c} B_\mu \left( \psi e^{i\epsilon \chi / \hbar c} \right)\]
so we get

\[ e^{ie\chi/\hbar c} B_{\mu}'(\psi) = B_{\mu}' \left( e^{-ie\chi/\hbar c} B_{\mu} \left( \psi e^{ie\chi/\hbar c} \right) e^{ie\chi/\hbar c} \right) \]

\[ = B_{\mu}' \left( B_{\mu} \left( \psi e^{ie\chi/\hbar c} \right) \right) \]

\[ = g_{\mu\nu} B_{\nu} \left( B_{\mu} \left( \psi e^{ie\chi/\hbar c} \right) \right) \]

(17) \hspace{1cm} (18) \hspace{1cm} (19)

In other words, replacing \( B \) and \( B' \) by their definitions, we have that the transformed K-G equation can be written as

\[ \left( p_{\mu} - \frac{e}{c} A_{\mu} \right) \left( p_{\mu} - \frac{e}{c} A_{\mu} \right) \psi = m_0 c^2 \psi \]

(20)

Thus applying a gauge transformation to the K-G equation just gives the same equation but with a wave function \( \psi' \) that is the original wave function \( \psi \) multiplied by a phase factor that depends only on \( \chi(x) \), and is the same for all wave functions. Since this phase factor cancels out in any calculation of an observable quantity, it doesn’t affect the physics and we can say that the K-G equation is invariant under gauge transformations.

We can extend the argument to higher powers the same way, but in summary, the result is that (using Greiner’s shorthand notation in his eqn 1.136):

\[ e^{ie\chi/\hbar c} \left( p_{\mu} - \frac{e}{c} A_{\mu} \right)^n = \left( p_{\mu} - \frac{e}{c} A_{\mu} \right)^n \left( \psi e^{ie\chi/\hbar c} \right) \]

(21)

For any function \( f \left( p_{\mu} - \frac{e}{c} A_{\mu} \right) \) that can be expanded in a power series, this result means that applying a gauge transformation to the function \( f \left( p_{\mu} - \frac{e}{c} A_{\mu} \right) \psi \) and get

\[ e^{ie\chi/\hbar c} f \left( p_{\mu} - \frac{e}{c} A_{\mu} \right) (\psi) = f \left( p_{\mu} - \frac{e}{c} A_{\mu} \right) \left( \psi e^{ie\chi/\hbar c} \right) \]

(22)