KLEIN-GORDON EQUATION WITH SCALAR 1/R POTENTIAL

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Reference: W. Greiner: Relativistic Quantum Mechanics (Wave Equations); 3rd Edition, Springer-Verlag (2000); Section 1.11, Exercise 1.16.

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In this exercise, Greiner introduces an unusual form of the Klein-Gordon equation, in which the potential is added to the $m_0^2 c^4$ term rather than modifying the four-momentum term. He uses a scalar interaction of the form

$$W(r) = -\frac{Z\alpha}{r}$$

We’ve already looked at the K-G equation for a particle in an electromagnetic field, where the interaction was introduced by modifying the four-momentum according to

$$p^\mu \rightarrow p^\mu - \frac{e}{c} A^\mu$$

This is as a result of the Hamiltonian for the electromagnetic force being given by

$$H = \frac{|p - eA/c|^2}{2m} + e\phi$$

which, when translated into four-vector notation results in the replacement given by [2]. This is known as minimal coupling of the momentum to the four-potential $A^\mu$.

In this case, we are coupling a scalar interaction given in general by $U(r)$ to the square of the mass. We start with the 3-d radial K-G equation that we had when considering the Coulomb potential:

$$\left[ -\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{l(l+1)}{r^2} \right] u(r) = \frac{(\varepsilon - V)^2 - m_0^2 c^4}{\hbar^2 c^2} u(r)$$

This equation includes the electromagnetic coupling for a Coulomb potential in the $(\varepsilon - V)^2$ factor. In our case, we remove this coupling and insert instead a coupling of the scalar interaction to the $m_0^2 c^4$ term. Removing the electromagnetic coupling gives
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\[
-\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{l(l+1)}{r^2} u(r) = \frac{\varepsilon^2 - m_0^2 c^4}{\hbar^2 c^2} u(r) \tag{5}
\]

and coupling to the $m_0^2 c^4$ term gives

\[
-\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{l(l+1)}{r^2} u(r) = \frac{\varepsilon^2 - m_0^2 c^4 - U^2(r)}{\hbar^2 c^2} u(r) \tag{6}
\]

After the usual substitution

\[ u(r) = \frac{R(r)}{r} \tag{7} \]

we get Greiner’s equation (2):

\[
\left[ \frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} + \frac{\varepsilon^2}{\hbar^2 c^2} - \frac{m_0^2 c^4}{\hbar^2 c^2} - \frac{U^2(r)}{\hbar^2 c^2} \right] R(r) = 0 \tag{8}
\]

Greiner now introduces a couple of dimensionless parameters to simplify things. First we have

\[ r' \equiv r \frac{\hbar c}{m_0 c^2} \tag{9} \]

Inserting this into \ref{eq:5} we get, after multiplying through by $\hbar^2 c^2 / m_0^2 c^4$:

\[
\left[ \frac{d^2}{dr'^2} - \frac{l(l+1)}{r'^2} + \frac{\varepsilon^2}{m_0^2 c^4} - 1 - \frac{U^2(r')}{m_0^2 c^4} \right] R(r') = 0 \tag{10}
\]

Note that all the terms in the square brackets are now dimensionless.

After another couple of parameters are introduced:

\[ b^2 \equiv 1 - \frac{\varepsilon^2}{m_0^2 c^4} \tag{11} \]

\[ d \equiv \frac{Z\alpha}{2b} \tag{12} \]

\[ \rho \equiv 2br' \tag{13} \]

we transform \ref{eq:10} into

\[
\left[ \frac{d^2}{d\rho^2} - \frac{l(l+1)}{\rho^2} - \frac{1}{4} + \frac{d}{\rho} \right] R(\rho) = 0 \tag{14}
\]

By examining the asymptotic behaviour for $\rho \to 0$ and $\rho \to \infty$, Greiner arrives at his eqn (15):

\[ R(\rho) = N\rho^{l+1}e^{-\rho/2}F(\rho) \tag{15} \]

Greiner has $r'$ and $r$ the wrong way round in his eqn (3).

I use $d = \frac{Z\alpha}{2b}$ rather than Greiner’s $c$ in order to avoid confusion with the speed of light.

The last term in Greiner’s eqn (9) should be $c/\rho$, not $\rho/c$. 
where $N$ is a normalization constant and $F$ is a function to be determined. By inserting this into 14 and calculating the derivatives (I checked this using Maple; the calculation gets a bit messy so I’ll just quote the result) we get

$$\rho \frac{d^2 F}{d\rho^2} + (2l + 2 - \rho) \frac{dF}{d\rho} + (d - l - 1) F = 0$$  \hspace{1cm} (16)

The solution is a confluent hypergeometric function

$$F(\rho) = {}_1 F_1 (l + 1 - d, 2l + 2, \rho)$$  \hspace{1cm} (17)

This function blows up at large $\rho$ at a rate that cannot be compensated by the $e^{-\rho/2}$ in 15 unless its first argument is a negative integer or zero (this condition results from the series form of ${}_1 F_1$ which must terminate). We therefore get the quantization condition

$$l + 1 - d = -n_r$$  \hspace{1cm} (18)

where $n_r$ is a positive integer. Defining the principal quantum number $n$ as

$$n \equiv l + 1 + n_r$$  \hspace{1cm} (19)

we can find the energy from 11 and 12

$$d = \frac{Z\alpha}{2\sqrt{1 - \frac{\varepsilon^2}{m_0^2c^4}}} = n$$  \hspace{1cm} (20)

which gives the energies

$$\varepsilon = \pm \sqrt{1 - \frac{Z^2\alpha^2}{4n^2} m_0 c^2}$$  \hspace{1cm} (21)

The energy doesn’t depend on the orbital angular momentum number $l$, but from 19 we see that since $n_r$ must be a positive integer, once we specify the principal quantum number $n$, then $l + n_r = n - 1$, so $l = 0, 1, \ldots, n - 1$ just as in the hydrogen atom from the Schrödinger equation.