ELECTROSTATICS - LINEAR CHARGE DISTRIBUTIONS

When faced with a continuous distribution of charge, we can work out the electric field as a function of position by using integration instead of summation. In general, we have

\[ E(r) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(r')}{|r - r'|^2} \frac{r - r'}{|r - r'|} \, d^3r' \]  

Here \( r' \) is the position of volume element \( d^3r' \) and \( \rho(r') \) is the charge density at that point. There are three types of problems that occur commonly with continuous charge distributions: linear, surface and volume charges. We’ll do a few examples using linear charges here to see how this works in practice. In many problems in electrostatics, it’s advisable to make use of any symmetries that the configuration has.

**Example 1**

Suppose we have a line segment extending along the \( x \) axis from \(-L\) to \( L\). This line segment contains a constant linear charge density \( \lambda \) (measured in Coulombs/metre). What is the electric field at a point \( z \) on the \( z \) axis?

We can split the problem in two by solving for the \( x \) and \( z \) components of \( E \) separately. Because the \( z \) axis divides the linear charge precisely in two, we can use symmetry to conclude that there is no net \( x \) component in the field. To work out the \( z \) component, we note that the contributions from \(+x\) and \(-x\) are equal.

In the formula above, \( r - r' \) is the vector from a point on the linear charge to the point \( z \) so we get

\[ |r - r'| = \sqrt{x^2 + z^2} \]  

The unit vector \( \frac{r - r'}{|r - r'|} \) has a \( z \) component of \( z/\sqrt{x^2 + z^2} \) and the charge density is \( \rho = \lambda \) so we get for the magnitude of \( E \) in the \( z \) direction:
\[ E_z = \frac{1}{4\pi\epsilon_0} \int_0^L \frac{2\lambda z}{(x^2 + z^2)^{3/2}} \, dx \]  \hspace{1cm} (3)

\[ = \frac{1}{4\pi\epsilon_0} \frac{2\lambda L}{z \sqrt{L^2 + z^2}} \]  \hspace{1cm} (4)

where you can either work out the integral by hand or look it up or use software like Maple.

Note that for \( z \gg L \), we get

\[ E_z \to \frac{1}{4\pi\epsilon_0} \frac{2\lambda L}{z^2} \]  \hspace{1cm} (5)

which is equivalent to the field of a point charge \( q = 2\lambda L \) at a distance \( z \).

For \( L \gg z \) we get

\[ E_z \to \frac{1}{4\pi\epsilon_0} \frac{2\lambda}{z} \]  \hspace{1cm} (6)

which is the formula for the field due to an infinitely long line of charge.

**Example 2**

A slight variant on this problem is to remove one half of the line segment, so the linear charge now extends from \( x = 0 \) to \( x = L \), with the test point still on the \( z \) axis. The \( z \) component of the field will now be half that calculated above:

\[ E_z = \frac{1}{4\pi\epsilon_0} \frac{\lambda L}{z \sqrt{L^2 + z^2}} \]  \hspace{1cm} (7)

However, since the problem is no longer symmetric about the origin, \( E_x \) is no longer zero. The \( x \) component of \( \frac{r - r'}{|r - r'|} \) is \(-x/\sqrt{x^2 + z^2}\) (the negative sign arises because the vector points from \( r' \) to \( r \), and since all \( r' \) locations are on the \(+x\) axis, and \( r \) is on the \( z \) axis, the \( x \) component of \( r - r' \) is always negative) so we get

\[ E_x = -\frac{1}{4\pi\epsilon_0} \int_0^L \frac{\lambda x}{(x^2 + z^2)^{3/2}} \, dx \]  \hspace{1cm} (8)

\[ = \frac{\lambda}{4\pi\epsilon_0} \frac{z - \sqrt{L^2 + z^2}}{z \sqrt{L^2 + z^2}} \]  \hspace{1cm} (9)

For \( z \gg L \):
Example 3

We now have a square loop (like a wire bent into a square) lying in the $xy$ plane with sides parallel to the axes and centred at the origin, and we want to find the field at some point $z$ on the $z$ axis. By symmetry, the field will be entirely along the $z$ direction, and the contributions from all four sides will be equal. If we consider the edge where $y = a/2$, then the distance from a point on this edge to $z$ is $\sqrt{x^2 + a^2 + z^2}$. Using the same reasoning as in example 1, the $z$ component of the unit vector connecting this point with $z$ is $z/\sqrt{x^2 + a^2 + z^2}$ so we get

$$E_z \rightarrow \frac{1}{4\pi \epsilon_0} \frac{\lambda}{z^2} \tag{10}$$

$$E_x \rightarrow 0 \tag{11}$$

As a check, when $z \gg a$, we get

$$E_z \rightarrow \frac{1}{4\pi \epsilon_0} \frac{4\lambda a}{z^2} \tag{14}$$

which is the equivalent of the field due to a point charge $q = 4\lambda a$.

Example 4

Now consider a circular loop of radius $r$ in the $xy$ plane, centred at the origin. By symmetry, the field is entirely in the $z$ direction. A line segment on the circle has length $rd\theta$, where $\theta$ is the angle in the $xy$ plane. The contribution from all line segments is the same. The $z$ component of the unit vector is $z/\sqrt{z^2 + r^2}$ so the field is

$$E_z = \frac{1}{4\pi \epsilon_0} \lambda rz \left(\frac{1}{z^2 + r^2 + z^2}\right)^{3/2} d\theta \tag{15}$$

$$E_z \rightarrow \frac{1}{4\pi \epsilon_0} \frac{2\pi r \lambda z}{(z^2 + r^2)^{3/2}} \tag{16}$$
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