METHOD OF IMAGES: TWO CONDUCTING PLANES

Another example of the method of images is the problem of a point charge $+q$ located at point $(x, y) = (a, b)$ in the first quadrant at $z = 0$, between two conducting planes that cover the $xz$ and $yz$ planes, thus these two conducting planes meet at right angles.

Following the procedure for the simpler problem of an image charge next to a single conducting plane, we can first place images of $-q$ at locations $(x, y) = (-a, b)$ and $(a, -b)$. If we stopped there, the potential would be

$$V = \frac{q}{4\pi \varepsilon_0} \left[ \frac{1}{\sqrt{(x-a)^2 + (y-b)^2 + z^2}} - \frac{1}{\sqrt{(x+a)^2 + (y-b)^2 + z^2}} - \frac{1}{\sqrt{(x-a)^2 + (y+b)^2 + z^2}} \right]$$

(1)

This clearly isn’t zero on either of the conducting planes, so we need to add another image. If we try an image of $+q$ at $(x, y) = (-a, -b)$ then the potential is

$$V = \frac{q}{4\pi \varepsilon_0} \left[ \frac{1}{\sqrt{(x-a)^2 + (y-b)^2 + z^2}} - \frac{1}{\sqrt{(x+a)^2 + (y-b)^2 + z^2}} - \frac{1}{\sqrt{(x-a)^2 + (y+b)^2 + z^2}} + \frac{1}{\sqrt{(x+a)^2 + (y+b)^2 + z^2}} \right]$$

(2)

Now $V = 0$ on both the planes $x = 0$ and $y = 0$ as can be seen by direct substitution.

The force on the original charge can be found from the force from the 3 images:

$$\mathbf{F} = \frac{q^2}{4\pi \varepsilon_0} \left( -\frac{1}{4a^2} \hat{x} - \frac{1}{4b^2} \hat{y} + \frac{1}{4(a^2 + b^2)^{3/2}} (a\hat{x} + b\hat{y}) \right)$$

(3)

The third term is the force between the actual charge and the image at $(x, y) = (-a, -b)$. The magnitude of this force is $q^2 / \left[ 4\pi \varepsilon_0 \left( 4a^2 + 4b^2 \right) \right]$ and we’ve resolved this along the two coordinate axes.
The work required to bring in $q$ from infinity can be found from the general formula for work applied to the images:

\[ W = \frac{1}{2} \sum_{i=1}^{n} q_i V(r_i) \]  

(4)

where $V(r_i)$ is the potential due to all the charges in the collection except $q_i$. However, this work assumes that we’re looking at all space, whereas the conducting planes cut the space under consideration down to a quarter of all space, so we need to divide the result by 4.

Plugging in the values we get

\[ V(a,b) = \frac{q}{4\pi\varepsilon_0} \left[ -\frac{1}{2a} - \frac{1}{2b} + \frac{1}{2\sqrt{a^2 + b^2}} \right] \]  

(5)

\[ V(-a,b) = \frac{q}{4\pi\varepsilon_0} \left[ \frac{1}{2a} + \frac{1}{2b} - \frac{1}{2\sqrt{a^2 + b^2}} \right] \]  

(6)

\[ V(-a,-b) = \frac{q}{4\pi\varepsilon_0} \left[ -\frac{1}{2a} - \frac{1}{2b} + \frac{1}{2\sqrt{a^2 + b^2}} \right] \]  

(7)

\[ V(a,-b) = \frac{q}{4\pi\varepsilon_0} \left[ \frac{1}{2a} + \frac{1}{2b} - \frac{1}{2\sqrt{a^2 + b^2}} \right] \]  

(8)

Thus

\[ W = \frac{1}{8\pi\varepsilon_0} \left[ \frac{q^2}{4\pi\varepsilon_0} \left[ -\frac{1}{2a} - \frac{1}{2b} + \frac{1}{2\sqrt{a^2 + b^2}} \right] \right] \]  

(9)

\[ = \frac{q^2}{8\pi\varepsilon_0} \left[ -\frac{1}{2a} - \frac{1}{2b} + \frac{1}{2\sqrt{a^2 + b^2}} \right] \]  

(10)

This technique can actually be applied to a configuration where we have two conducting planes meeting at other angles. Since the image charges have to cancel in pairs, we can derive a formula for the case where we can divide up space into an even number $n$ of sectors. We’ve just seen how it works for $n = 4$, but for general even $n$ the argument would go like this.

Suppose the sector bounded by the planes has one plane at $y = 0$ (that is, the $xz$ plane) and the other at an angle of $2\pi/n$. Let’s place the test charge at a location $(x, y) = d(\cos\alpha, \sin\alpha)$ where $\alpha$ is the angle the radius vector to the charge makes with the plane $y = 0$ and $d$ is the distance of the charge from the origin. We can then place the first image by drawing a line from the charge perpendicular to the plane at angle $2\pi/n$ and extending it an equal distance on the other side. This image will have charge $-q$ and be at an angle of $2 \times \frac{2\pi}{n} - \alpha = \frac{4\pi}{n} - \alpha$. The next image is found by drawing
a line from the first image perpendicular to the plane at angle $4\pi/n$ and extending it an equal distance on the other side. This charge is $+q$ and is at angle $\frac{4\pi}{n} + \alpha$. We continue in this fashion until we arrive at a charge of $-q$ at angle $2\pi - \alpha$, which is the image of the original charge in the $y = 0$ plane. This will give a total of $n$ charges (one is the test charge and the other $n - 1$ are the images). The potential is then

\[
\frac{4\pi \epsilon_0}{q} V = \sum_{m=0}^{n-1} \frac{1}{\sqrt{(x - d \cos (\frac{4\pi m}{n} + \alpha))^2 + (y - d \sin (\frac{4\pi m}{n} + \alpha))^2 + z^2}} - \sum_{m=1}^{n} \frac{1}{\sqrt{(x - d \cos (\frac{4\pi m}{n} - \alpha))^2 + (y - d \sin (\frac{4\pi m}{n} - \alpha))^2 + z^2}}
\]

(11)

(12)