LAPLACE’S EQUATION - CHARGED DISK

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In Example 3 in an earlier post, we found that the potential of a uniformly charged disk of radius $R$ can be worked out for points on the axis of the disk and is

$$ V(z) = \frac{\sigma}{2\varepsilon_0} \left( \sqrt{z^2 + R^2} - |z| \right) \quad (1) $$

In spherical coordinates, a point with coordinate $z$ on the axis has coordinates $(r, 0)$ if $z > 0$ or $(r, \pi)$ if $z < 0$. We can therefore write

$$ V(r, 0) = V(r, \pi) = \frac{\sigma}{2\varepsilon_0} \left( \sqrt{r^2 + R^2} - r \right) \quad (2) $$

This formula is valid only for points on the $z$ axis, but we can use it, together with the general solution of Laplace’s equation in terms of Legendre polynomials, to get an approximation for the potential off the $z$ axis.

We consider first the case of $r > R$. In this case the most general solution of Laplace’s equation in spherical coordinates is:

$$ V(r, \theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta) \quad (3) $$

For $\theta = 0$ we get, since $P_l(0) = 1$ for all $l$:

$$ V(r, 0) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} \quad (4) $$

To find the coefficients $B_l$ we need to expand $V(r, 0)$ in powers of $(R/r)$:

$$ V(r, 0) = \frac{\sigma r}{2\varepsilon_0} \left( \sqrt{1 + \frac{R^2}{r^2}} - 1 \right) \quad (5) $$

$$ = \frac{\sigma}{2\varepsilon_0} \left( \frac{R^2}{2r} - \frac{R^4}{8r^3} + \ldots \right) \quad (6) $$
Comparing with 4, we get

\[ B_0 = \frac{\sigma R^2}{4\epsilon_0} \quad (7) \]
\[ B_1 = 0 \quad (8) \]
\[ B_2 = -\frac{\sigma R^4}{16\epsilon_0} \quad (9) \]

so the general formula for the potential off the axis is

\[ V(r, \theta) = \frac{\sigma}{2\epsilon_0} \left[ \frac{R^2}{2r} P_0(\cos \theta) - \frac{R^4}{8r^3} P_2(\cos \theta) + \ldots \right] \quad (10) \]

Since the odd polynomial terms all vanish, and the even Legendre polynomials are even functions, this expansion is also valid for the region \( z < 0 \), where \( \pi/2 < \theta \leq \pi \).

For the case \( r < R \), we have

\[ V(r, \theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos \theta) \quad (11) \]

When \( \theta = 0 \), we get

\[ V(r, 0) = \sum_{l=0}^{\infty} A_l r^l \quad (12) \]

We can now expand 2 in powers of \( r/R \), and we get

\[ V(r, 0) = \frac{\sigma}{2\epsilon_0} \left( R \sqrt{1 + \frac{r^2}{R^2}} - r \right) \]
\[ = \frac{\sigma}{2\epsilon_0} \left( R - r + \frac{r^2}{2R} - \frac{r^4}{8R^3} + \ldots \right) \quad (14) \]

Comparing this to 12, we get

\[ A_0 = \frac{\sigma R}{2\epsilon_0} \quad (15) \]
\[ A_1 = -\frac{\sigma}{2\epsilon_0} \quad (16) \]
\[ A_2 = \frac{\sigma}{4\epsilon_0 R} \quad (17) \]
\[ A_3 = 0 \quad (18) \]

The off-axis expansion thus begins:
\[ V(r, \theta) = \frac{\sigma}{2\epsilon_0} \left[ RP_0(\cos \theta) - rP_1(\cos \theta) + \frac{r^2}{2R} P_2(\cos \theta) + \ldots \right] \] (19)

In this case, since there is a \( P_1 \) term, the same expansion isn’t valid for \( \theta = \pi \). However, because the odd polynomials are odd functions, for \( \theta = \pi \) we have

\[ V(r, \pi) = \sum_{l=0}^{\infty} (-1)^l A_l r^l \] (20)

The only change we need to make in the expansion is thus to change the sign of \( A_1 \) so we get

\[ V(r, \theta) = \frac{\sigma}{2\epsilon_0} \left[ RP_0(\cos \theta) + rP_1(\cos \theta) + \frac{r^2}{2R} P_2(\cos \theta) + \ldots \right] \] (21)

All higher odd polynomials have \( A_l = 0 \) so this is the only change that is required in the entire expansion.

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