As another example of applying the solution to Laplace’s equation in cylindrical coordinates, we consider the following problem. We are given a cylindrical non-conducting shell or radius \( R \) carrying a charge density of \( \sigma_0 \) on its upper half \((-\pi < \phi < 0)\) and \(-\sigma_0\) on its lower half \((0 < \phi < \pi)\).

Find the potential everywhere.

We can begin in the same manner as for the other problem involving a cylindrical shell that we solved earlier. The solution is the same up until the point where we introduce the surface charge.

Thus, outside the shell, we have

\[
V_{\text{out}} = B_{\text{out}} + \sum_{n=1}^{\infty} \left[ \frac{D_n}{r^n} \cos n\phi - \frac{C_n}{r^n} \sin n\phi \right]
\]  \hspace{1cm} (1)

Inside the shell, we have

\[
V_{\text{in}} = B_{\text{in}} + \sum_{n=1}^{\infty} \left[ A_n r^n \sin n\phi + B_n r^n \cos n\phi \right]
\]  \hspace{1cm} (2)

Since the potential is continuous over a surface charge, we must have \( V_{\text{out}}(R) = V_{\text{in}}(R) \), so we get

\[
B_{\text{out}} + \sum_{n=1}^{\infty} \left[ \frac{D_n}{R^n} \cos n\phi - \frac{C_n}{R^n} \sin n\phi \right] = B_{\text{in}} + \sum_{n=1}^{\infty} \left[ A_n R^n \sin n\phi + B_n R^n \cos n\phi \right]
\]  \hspace{1cm} (3)

Equating coefficients of the sine and cosine, we get

\[
B_{\text{out}} = B_{\text{in}} \hspace{1cm} (4)
\]

\[
C_n = -A_n R^{2n} \hspace{1cm} (5)
\]

\[
D_n = B_n R^{2n} \hspace{1cm} (6)
\]

The outward derivative of the potential is discontinuous across a surface charge, and we have
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$$\left. \frac{\partial V}{\partial r} \right|_{out} - \left. \frac{\partial V}{\partial r} \right|_{in} = -\frac{\sigma}{\epsilon_0}$$  \hspace{1cm} (7)

Plugging in the formulas for $V_{out}$ and $V_{in}$, we get

$$\sum_{n=1}^{\infty} \left[ -\frac{nR^{2n}A_n}{R^{n+1}} - nR^{n-1}A_n \right] \sin n\phi + \sum_{n=1}^{\infty} \left[ -\frac{nR^{2n}B_n}{R^{n+1}} - nR^{n-1}B_n \right] \cos n\phi = \begin{cases} \frac{\sigma_0}{\epsilon_0} & -\pi < \phi < 0 \\ -\frac{\sigma_0}{\epsilon_0} & 0 < \phi < \pi \end{cases}$$  \hspace{1cm} (8)

Since the surface charge is an odd function of $\phi$, we can eliminate the cosine terms, since the cosine is an even function. Therefore, we have $B_n = D_n = 0$. We are free to choose the potential at infinity to be any constant, so we might as well take it to be zero, in which case we have $B_{in} = B_{out} = 0$. We are therefore left with, after simplifying the term in brackets:

$$-2 \sum_{n=1}^{\infty} nR^{n-1}A_n \sin n\phi = \begin{cases} \frac{\sigma_0}{\epsilon_0} & -\pi < \phi < 0 \\ -\frac{\sigma_0}{\epsilon_0} & 0 < \phi < \pi \end{cases}$$  \hspace{1cm} (9)

To find the $A_n$, we can use the fact that the set of $\sin n\phi$ functions is orthogonal over the interval $[-\pi, \pi]$. That is

$$\int_{-\pi}^{\pi} \sin m\phi \sin n\phi d\phi = \begin{cases} 0 & n \neq m \\ \pi & n = m \end{cases}$$  \hspace{1cm} (10)

Therefore we can multiply both sides by $\sin m\phi$ and integrate to get

$$-2\pi mR^{m-1}A_m = \frac{\sigma_0}{\epsilon_0} \left[ \int_{-\pi}^{0} \sin m\phi d\phi - \int_{0}^{\pi} \sin m\phi d\phi \right]$$  \hspace{1cm} (11)

The integrals in brackets on the right come out to

$$\left[ \int_{-\pi}^{0} \sin m\phi d\phi - \int_{0}^{\pi} \sin m\phi d\phi \right] = \begin{cases} 0 & m \text{ even} \\ -\frac{4}{m} & m \text{ odd} \end{cases}$$  \hspace{1cm} (12)

Therefore (changing the index from $m$ to $n$ for convenience):

$$A_n = \begin{cases} 0 & n \text{ even} \\ \frac{2\sigma_0}{\pi\epsilon_0 n^2R^{n-1}} & n \text{ odd} \end{cases}$$  \hspace{1cm} (13)

We thus get

$$C_n = \begin{cases} 0 & n \text{ even} \\ -\frac{2\sigma_0 R^{n+1}}{\pi\epsilon_0 n^2} & n \text{ odd} \end{cases}$$  \hspace{1cm} (14)

The final formula for the potential is
\[ V(r, \phi) = \begin{cases} 
\frac{2\sigma_0}{\pi \epsilon_0} \sum_{n \text{ odd}} \frac{r^n}{n^2 R^{n-1}} \sin n\phi & r < R \\
\frac{2\sigma_0}{\pi \epsilon_0} \sum_{n \text{ odd}} \frac{R^{n+1}}{n^2 r^n} \sin n\phi & r > R 
\end{cases} \]  
(15)