

GREEN'S RECIPROCITY THEOREM

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problems 3.43 & 3.44.

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There is an interesting theorem that relates two separate charge distributions. Suppose we have a charge distribution ρ_1 with its associated potential V_1 , and a completely separate charge distribution ρ_2 with potential V_2 . These two distributions do not co-exist; they are completely separate situations.

Now consider the electric fields \mathbf{E}_1 and \mathbf{E}_2 produced by these two distributions. We can consider the following integral, taken over all space:

$$\int \mathbf{E}_1 \cdot \mathbf{E}_2 d^3 \mathbf{r} = - \int \nabla V_1 \cdot \mathbf{E}_2 d^3 \mathbf{r} \quad (1)$$

Let's consider the first term in the dot product, and use integration by parts:

$$- \int \frac{\partial V_1}{\partial x} E_{2x} dx dy dz = - \int V_1 E_{2x} |_{all\ x} dy dz + \int V_1 \frac{\partial E_{2x}}{\partial x} dx dy dz \quad (2)$$

If we make the usual assumption that the potential V_1 vanishes at infinity then the integrated term is zero. Doing similar integrals for the other terms in the dot product (integrating with respect to y and then z first for the second and third terms respectively) gives us:

$$- \int \nabla V_1 \cdot \mathbf{E}_2 d^3 \mathbf{r} = \int V_1 \nabla \cdot \mathbf{E}_2 d^3 \mathbf{r} \quad (3)$$

$$= \frac{1}{\epsilon_0} \int V_1 \rho_2 d^3 \mathbf{r} \quad (4)$$

where in the last line we used Gauss's law relating the field to the charge distribution.

We could just as well have done the same calculation interchanging the subscripts 1 and 2, so we get

$$\int V_1 \rho_2 d^3 \mathbf{r} = \int V_2 \rho_1 d^3 \mathbf{r} \quad (5)$$

which is *Green's reciprocity theorem*.

Now for a few examples of its use.

Example 1. Suppose we have two conductors, each of which can be of arbitrary shape and location. First, we place a charge Q on conductor 1, which induces a potential V_{12} on conductor 2 (which has no net charge). Second, we do the opposite: we place the same charge Q on conductor 2, which induces a potential V_{21} on conductor 1 (which now has no net charge).

Since the potential on a conductor is always constant, we can use the reciprocity theorem to say

$$V_{12} \int \rho_2 d^3 \mathbf{r} = V_{21} \int \rho_1 d^3 \mathbf{r} \quad (6)$$

The two integrals are the same, and give the total charge Q , so we can conclude that $V_{12} = V_{21}$. Note that this result does not depend on the shape or location of the conductors; it's a universal result.

Example 2. We have two parallel infinite conducting planes, both of which are grounded. The distance between the plates is d . We place a point charge q between the plates at a distance x from plate 1 (which we take to be the left plate). Find the total charge induced on each plate.

To apply the reciprocity theorem, we need two distinct charge distributions. For the first, we can take the system as described. For the second, we can remove the charge q and also remove the condition that the plates are grounded, so each plate can be at a different potential.

First, consider the distribution as given. Let the potential of the left plate be V_l and of the right plate be V_r . Since the two plates are grounded, we have $V_l = V_r = 0$. Also, since the plates are grounded, the induced charge must cancel out the point charge so there is zero net charge in the system. That is $Q_l + Q_r = -q$.

Now consider the distribution without the point charge q . In this case we can take the potential of the left plate to be $V'_l = 0$ and of the right plate to be $V'_r = V_0$. Note that this assumes the right plate is not grounded. This doesn't matter, since the second distribution can be anything we like. We assume that the plates here have total charges Q'_l and Q'_r , although we'll see we don't need these values anyway.

Since the second distribution contains no point charge, the potential varies linearly between the two plates, so the potential at position x is $V'_x = V'_l + V_0 x/d = V_0 x/d$. Now we're ready to apply the reciprocity theorem. The

charge density ρ_2 consists only of the charge on the two plates, since we've removed q . We have, on one side:

$$\int V_1 \rho_2 d^3 \mathbf{r} = V_l Q'_l + V_r Q'_r \quad (7)$$

$$= 0 \quad (8)$$

since $V_l = V_r = 0$.

On the other side, we have

$$\int V_2 \rho_1 d^3 \mathbf{r} = V'_l Q_l + V'_x q + V'_r Q_r \quad (9)$$

$$= V_0 \left(\frac{qx}{d} + Q_r \right) \quad (10)$$

From the theorem, this must be zero, so we get

$$Q_r = -\frac{qx}{d} \quad (11)$$

$$Q_l = -q + \frac{qx}{d} \quad (12)$$

$$= -q \left(1 - \frac{x}{d} \right) \quad (13)$$

Note that the reciprocity theorem in this case allows us to calculate only the *total* charge on each plate; finding the actual surface charge density is a considerably harder problem.

Example 3. We have two concentric spherical conductors of radii a and $b > a$, and a point charge q between them at a location r such that $a < r < b$. Again assuming the spheres are grounded, find the total induced charge on each sphere.

Using similar notation to the last example, we again consider the two distributions to be the original configuration (with the charge q) and a configuration without q . In the first case, since the conductors are grounded, we have

$$V_a = V_b = 0 \quad (14)$$

In the second case, we can take $V'_a = V_0$. In this case, since we don't have the charge q , the system has spherical symmetry, so any charge distributed over the spheres must be uniform, so the potential and the field are the same as if the charge were concentrated at the centre of the spheres. This means

that the potential between the spheres has a $1/r$ dependence, so we can write

$$V'_a = V_0 \quad (15)$$

$$V'_r = \frac{a}{r}V_0 \quad (16)$$

$$V'_b = \frac{a}{b}V_0 \quad (17)$$

Applying the reciprocity theorem, we get

$$\int V_1 \rho_2 d^3 \mathbf{r} = V_a Q'_a + V_b Q'_b \quad (18)$$

$$= 0 \quad (19)$$

On the other side, we have

$$\int V_2 \rho_1 d^3 \mathbf{r} = V'_a Q_a + V'_r q + V'_b Q_b \quad (20)$$

$$= V_0 \left(Q_a + \frac{a}{r}q + \frac{a}{b}Q_b \right) \quad (21)$$

Again, since the two spheres are grounded in the first configuration, we must have $Q_a + Q_b = -q$, so we get

$$\int V_2 \rho_1 d^3 \mathbf{r} = V_0 \left(-q - Q_b + \frac{a}{r}q + \frac{a}{b}Q_b \right) \quad (22)$$

$$= 0 \quad (23)$$

Solving this for Q_b we get

$$Q_b = -q \frac{(r-a)b}{(b-a)r} \quad (24)$$

$$Q_a = -q - Q_b \quad (25)$$

$$= -q \frac{(b-r)a}{(b-a)r} \quad (26)$$