FIELD OF A POLARIZED CYLINDER

As another example of the use of the bound charges representation of a polarized object, we can look at a cylinder which contains a uniform polarization $P$ perpendicular to its axis. In this case, $\nabla \cdot P = 0$ everywhere inside the cylinder. If we take the direction of $P$ to be $\phi = 0$ then $P \cdot \hat{n} = P \cos \phi$. We thus must find the potential of an infinite cylinder with a surface charge of

$$\sigma_b = P \cos \phi \quad (1)$$

We’ve worked out the general solution to Laplace’s equation in cylindrical coordinates before, so we can use the results from there. The solution inside the cylinder is

$$V_{in} = B_{in} + \sum_{n=1}^{\infty} \left[ A_n r^n \sin n\phi + B_n r^n \cos n\phi \right] \quad (2)$$

while outside it is

$$V_{out} = B_{out} + \sum_{n=1}^{\infty} \left[ \frac{D_n}{r^n} \cos n\phi - \frac{C_n}{r^n} \sin n\phi \right] \quad (3)$$

From the boundary condition requiring the potential to be continuous at the surface of the cylinder we get the relations

$$B_{out} = B_{in} \quad (4)$$
$$C_n = -A_n R^{2n} \quad (5)$$
$$D_n = B_n R^{2n} \quad (6)$$

where $R$ is the radius of the cylinder.

From the condition on the derivative of the potential at the boundary, we get
\[
\sum_{n=1}^{\infty} \left[ 2nR^{n-1}A_n \right] \sin n\phi + \sum_{n=1}^{\infty} \left[ 2nR^{n-1}B_n \right] \cos n\phi = \frac{\sigma_b}{\epsilon_0} \quad (7)
\]

From (1) we see that all coefficients of the sine terms are zero, as are all coefficients of cosine terms except for \( n = 1 \). We therefore get

\[
B_1 = \frac{P}{2\epsilon_0} \quad (8)
\]
\[
D_1 = \frac{PR^2}{2\epsilon_0} \quad (9)
\]
\[
= \frac{PR^2}{2\epsilon_0} \quad (10)
\]

The potential in the two regions is thus

\[
V_{in} = \frac{P}{2\epsilon_0} r \cos \phi \quad (11)
\]
\[
V_{out} = \frac{PR^2}{2r\epsilon_0} \cos \phi \quad (12)
\]

From this we can get the field by taking the negative gradient. Since \( r \cos \phi = x \), the field inside the cylinder is

\[
E_{in} = -\nabla V_{in} \quad (13)
\]
\[
= -\frac{P}{2\epsilon_0} \hat{x} \quad (14)
\]

The interior field is thus uniform, just as the field inside a uniformly polarized sphere is uniform.

Outside the cylinder we need to take the gradient in cylindrical coordinates. We get

\[
E_{out} = -\frac{\partial V_{out}}{\partial r} \hat{r} - \frac{1}{r} \frac{\partial V_{out}}{\partial \phi} \hat{\phi} \quad (15)
\]
\[
= \frac{R^2}{2r^2\epsilon_0} \left[ P \cos \phi \hat{r} + P \sin \phi \hat{\phi} \right] \quad (16)
\]

We can write this in terms of the polarization vector. If \( \mathbf{P} \) points in the direction \( \phi = 0 \) then we have

\[
\mathbf{P} = P \cos \phi \hat{r} - P \sin \phi \hat{\phi} \quad (17)
\]
where the minus sign on the second term is because \( \hat{\phi} \) points in the direction of increasing \( \phi \), which is clockwise from \( \phi = 0 \). We have then

\[
P \cos \phi \hat{r} + P \sin \phi \hat{\phi} = 2(\mathbf{P} \cdot \hat{r}) \hat{r} - \mathbf{P}
\]  

(18)

so the field is

\[
\mathbf{E}_{\text{out}} = \frac{R^2}{2r^2 \epsilon_0} [2(\mathbf{P} \cdot \hat{r}) \hat{r} - \mathbf{P}]
\]  

(19)