BIOT-SAVART LAW - CURRENT LOOPS

As mentioned earlier, besides feeling a force from an external magnetic field, an electric current also produces its own magnetic field. The experimentally determined rule for calculating this generated magnetic field is known as the Biot-Savart law. For a steady current (one that doesn’t vary with time) in a wire, this law can be written as

$$B(r) = \frac{\mu_0}{4\pi} \int \frac{I \times (r - r')}{|r - r'|^3} dl'$$  \hspace{1cm} (1)

where $r'$ is a location on the wire and $r$ is the point at which you want to determine the magnetic field. The constant $\mu_0$ is known as the permeability of free space, and is the magnetic analogue to $\epsilon_0$ in electrostatics. Its value is

$$\mu_0 = 1.25663706 \times 10^{-6} \text{ m kg s}^{-2} \text{Amp}^{-2}$$ \hspace{1cm} (2)

Again, this law isn’t derived from anything more fundamental; it’s a generalization of experiment, although the value of $\mu_0$ is fixed at exactly $4\pi \times 10^{-7} \text{ m kg s}^{-2} \text{Amp}^{-2}$.

As an example, suppose we want to find the field generated by a steady current travelling round a square loop of side length $2R$, at the centre of the square. We can do this by finding the field generated by a single wire segment first.

Because of the cross product in the integrand, the field will be perpendicular to the plane of the square, so if we call this direction the $z$ axis, and set the edge of the square at $x = R$, we can write the integral as (setting $r = 0$ since we’re interested in the field at the origin):
\[ B_1(r) = \hat{z} \frac{I \mu_0}{4\pi} \int \frac{|r - r'| \sin \theta}{|r - r'|^3} dl' \quad (3) \]

\[ = \hat{z} \frac{I \mu_0}{4\pi} \int_{-R}^{R} \frac{R}{(R^2 + y^2)^{3/2}} dy \quad (4) \]

\[ = \hat{z} \frac{\sqrt{2} I \mu_0}{4\pi R} \quad (5) \]

where \( \theta \) is the angle between \( I \) and \( r - r' \), so that \( \sin \theta = R / |r - r'| \).

By symmetry, the contribution from all 4 sides is equal, so we get for the total field

\[ B = 4B_1 = \hat{z} \frac{\sqrt{2} I \mu_0}{\pi R} \quad (6) \]

Now suppose we have a current loop consisting of a regular polygon with \( n \) sides. In this case, each side subtends an angle of \( 2\pi / n \), so if we align one side parallel to the \( y \) axis at \( x = R \), this side will extend from an angle of \( -\pi / n \) to \( +\pi / n \), and will have a length of \( 2R \tan \frac{\pi}{n} \). Now the integral for a single side is

\[ B_1(0) = \hat{z} \frac{I \mu_0}{4\pi} \int_{-R \tan \frac{\pi}{n}}^{R \tan \frac{\pi}{n}} \frac{R}{(R^2 + y^2)^{3/2}} dy \quad (7) \]

\[ = \hat{z} \frac{I \mu_0}{4\pi} \frac{2R \tan \frac{\pi}{n}}{R \sqrt{R^2 + (R \tan \frac{\pi}{n})^2}} \quad (8) \]

\[ = \hat{z} \frac{I \mu_0}{2\pi R} \frac{\sin \frac{\pi}{n}}{n} \quad (9) \]

Each of the \( n \) sides will still contribute an equal amount, so the total field is

\[ B = nB_1 = \hat{z} \frac{nI \mu_0}{2\pi R} \frac{\sin \frac{\pi}{n}}{n} \quad (10) \]

As \( n \to \infty \), this formula should give us the field due to a circular loop.

In this limit, we can approximate the sine by the first term in its Taylor expansion \( \sin \frac{\pi}{n} \approx \frac{\pi}{n} \) so we get

\[ \lim_{n \to \infty} B = \hat{z} \frac{I \mu_0}{2R} \quad (11) \]
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