MAGNETIC FIELD OF A SOLENOID

A solenoid is a helical coil of wire wound round an insulating cylinder. We can find the magnetic field due to a solenoid carrying a steady current \( I \) as follows. First, we work out the field due to a single circular loop of radius \( a \) as measured at a point \( P \) on the axis (which we’ll take to be the \( z \) axis) of the loop. The diagram below shows a side view of the solenoid.

In the diagram, we’ll take the circular loop to be the first loop in the solenoid (closest to \( P \)), whose radius subtends an angle \( \theta_1 \) at \( P \). The distance from \( P \) along the axis to the centre of the circle is \( z \). (We’ll refer to the other parts of the diagram later.)

By symmetry, the components of the field in the \( x \) and \( y \) directions cancel, so we need calculate only the \( z \) component. The Biot-Savart law in this case is

\[
B(r) = \hat{z} \frac{I \mu_0}{4\pi} \int \frac{dl \times (r - r')}{|r - r'|^3}
\]  

The line element \( dl \) is tangent to the circle, and the vector \( r - r' \) is always perpendicular to it. Look at the point on the loop where the tail of the \( \theta_1 \) vector meets the current loop, and assume the current is coming out of the page at this point. Then applying the right-hand rule, we see that the field vector lies in the plane of the figure, and points diagonally to the upper right, making an angle of \( \frac{\pi}{2} - \theta_1 \) with the \( z \) axis. To project out the \( z \) component

\[
\text{in the diagram, we’ll take the circular loop to be the first loop in the solenoid (closest to } P \text{), whose radius subtends an angle } \theta_1 \text{ at } P. \text{ The distance from } P \text{ along the axis to the centre of the circle is } z. \text{(We’ll refer to the other parts of the diagram later.)}

By symmetry, the components of the field in the } x \text{ and } y \text{ directions cancel, so we need calculate only the } z \text{ component. The Biot-Savart law in this case is}

\[
B(r) = \hat{z} \frac{I \mu_0}{4\pi} \int \frac{dl \times (r - r')}{|r - r'|^3}
\]  

The line element } dl \text{ is tangent to the circle, and the vector } r - r' \text{ is always perpendicular to it. Look at the point on the loop where the tail of the} \theta_1 \text{ vector meets the current loop, and assume the current is coming out of the page at this point. Then applying the right-hand rule, we see that the field vector lies in the plane of the figure, and points diagonally to the upper right, making an angle of } \frac{\pi}{2} - \theta_1 \text{ with the } z \text{ axis. To project out the } z \text{ component}
of the field, we multiply by \( \cos \left( \frac{\pi}{2} - \theta_1 \right) = \sin \theta_1 = \frac{a}{|r - r'|} \). This angle is a constant, as is \(|r - r'|\), and the integration extends round the circumference of the circle, so we get

\[
B_z = \frac{I \mu_0}{4\pi} \frac{a}{|r - r'|} \frac{2\pi a}{|r - r'|^2} 
\]

\[
= \frac{I \mu_0}{2a} \sin^3 \theta_1 
\]

Now suppose the solenoid has \( n \) turns per unit length, and that this number is large enough that we can approximate the solenoid by a circular surface current density of \( In \) per unit length, and thus use integration to determine the overall field. To do the integral, we need to work out relation between a change in \( \theta \) and a change in \( z \), as shown in the diagram. An increase of \( dz \) means a decrease in angle of \( -d\theta \) as shown. To work out the relation:

\[
\sin \theta = \frac{a}{\sqrt{a^2 + z^2}} \]

\[
\sin (\theta - d\theta) = \frac{a}{\sqrt{a^2 + (z + dz)^2}} 
\]

We can expand the second equation in a Taylor series and retain only first terms in the differentials, and we get

\[
\sin \theta - \cos \theta d\theta = \frac{a}{\sqrt{a^2 + z^2}} - \frac{az}{(a^2 + z^2)^{3/2}} dz 
\]

Replacing the terms on the right by trig functions, we get

\[
\frac{a}{\sqrt{a^2 + z^2}} - \frac{az}{(a^2 + z^2)^{3/2}} dz = \sin \theta - \frac{1}{a} \cos \theta \sin^2 \theta dz 
\]

Thus we get

\[
dz = \frac{a}{\sin^2 \theta} d\theta 
\]

The current in an infinitesimal slice of the solenoid is therefore

\[
Indz = \frac{Ina}{\sin^2 \theta} d\theta 
\]

We can now find the total field by doing the integral
MAGNETIC FIELD OF A SOLENOID

\[ B_z = \int_{\theta_2}^{\theta_1} \frac{\mu_0}{4\pi a} \frac{\sin^3 \theta}{\sin^2 \theta} \frac{Ina}{\sin^2 \theta} d\theta \]

\[ = \frac{n\mu_0 I}{2} \left( \cos \theta_2 - \cos \theta_1 \right) \]

This might look like it’s independent of the radius \( a \), but this dependence is included in the angles.

For an infinite solenoid, \( \theta_1 \to \pi \) and \( \theta_2 \to 0 \) and we get

\[ B_\infty = n\mu_0 I \]

PINGBACKS

Pingback: Solenoid with arbitrary cross-section
Pingback: Magnet falling through a metal pipe
Pingback: Faraday’s law, Ampère’s law and the quasistatic approximation
Pingback: Self-inductance
Pingback: Energy in a magnetic field
Pingback: Angular momentum in electromagnetic fields
Pingback: Energy transfer in a solenoid
Pingback: Angular momentum conservation: example with a solenoid
Pingback: Momentum of a point charge outside a solenoid
Pingback: Magnetic systems in thermodynamics