MAGNETIC VECTOR POTENTIAL AS THE CURL OF ANOTHER FUNCTION

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Reference: Griffiths, David J. (2007) Introduction to Electrodynamics, 3rd Edition; Prentice Hall - Problem 5.53.

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The fact that $\mathbf{B}$ is divergenceless allows us to express it as the curl of a vector potential: $\mathbf{B}=\nabla\times\mathbf{A}$. By assuming $\nabla\cdot\mathbf{A}=0$ as well, can express it in turn as the curl of another function: $\mathbf{A}=\nabla\times\mathbf{W}$. Although this is rarely useful, we can try to work out $\mathbf{W}$ in a few cases.

First, we can express $\mathbf{W}$ in terms of $\mathbf{B}$ by noting

$$\mathbf{B} = \nabla\times\mathbf{A} \tag{1}$$

$$= \nabla\times(\nabla\times\mathbf{W}) \tag{2}$$

$$= \nabla(\nabla\cdot\mathbf{W})-\nabla^{2}\mathbf{W} \tag{3}$$

If we assume that $\mathbf{W}$ is divergenceless as well, then we get

$$\nabla^{2}\mathbf{W}=-\mathbf{B} \tag{4}$$

This is the same form as the equation for the vector potential in terms of the current density:

$$\nabla^{2}\mathbf{A}=-\mu_{0}\mathbf{J} \tag{5}$$

This could be written in integral form as

$$\mathbf{A}(\mathbf{r})=\frac{\mu_{0}}{4\pi}\int_{V}\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^{3}\mathbf{r}' \tag{6}$$

Therefore we can write

$$\mathbf{W}(\mathbf{r})=\frac{1}{4\pi}\int_{V}\frac{\mathbf{B}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^{3}\mathbf{r}' \tag{7}$$

It's important to note that this equation relies on $\mathbf{B}\to 0$ at infinite distance, just as the equation for $\mathbf{A}$ relied on the current density being finite in extent.

We can now find $\mathbf{W}$ in a couple of special cases. First, consider the case of a constant field. We might be tempted to try calculating $\mathbf{W}$ by simply taking **B** outside the integral above, but this integral formula doesn't apply in this case since **B** extends to infinity. We can, however, use the result we got earlier for the vector potential for a constant field.

$$\mathbf{A} = -\frac{1}{2}\mathbf{r}\times\mathbf{B} \tag{8}$$

We need to solve the equation

$$\nabla\times\mathbf{W} = -\frac{1}{2}\mathbf{r}\times\mathbf{B} \tag{9}$$

One way of approaching this is to split the vector equation into its components. For the x component we have

$$\frac{\partial W_z}{\partial y} - \frac{\partial W_y}{\partial z} = -\frac{1}{2}\left(yB_z - zB_y\right) \tag{10}$$

We can try a solution of the form

$$\begin{aligned}
\frac{\partial W_z}{\partial y} &= -\frac{1}{2}yB_z && (11)\\
W_z &= -\frac{1}{4}y^2B_z + f(x,z) && (12)\\
\frac{\partial W_y}{\partial z} &= -\frac{1}{2}zB_y && (13)\\
W_y &= -\frac{1}{4}z^2B_y + g(x,y) && (14)
\end{aligned}$$

where f and g are functions of integration.

Working out the z component, we get

$$W_y = -\frac{1}{4}x^2B_y + h(y,z) \tag{15}$$

Comparing the two equations for W_y we get

$$W_y = -\frac{1}{4}B_y\left(x^2+z^2\right) \tag{16}$$

The other two components can be worked out the same way and we get

$$\mathbf{W} = -\frac{1}{4}\left[B_x\left(y^2+z^2\right)\hat{\mathbf{x}} + B_y\left(x^2+z^2\right)\hat{\mathbf{y}} + B_z\left(x^2+y^2\right)\hat{\mathbf{z}}\right] \tag{17}$$

By direct calculation we can check that $\nabla\cdot\mathbf{W} = 0$ so this is a valid solution. (There may be some fancy way of expressing this entirely in terms of vectors and their products, but if so, it eluded me.) The solution is unlikely to be unique.

For a second example, we can return to the infinite solenoid. Griffiths shows in his Example 5.12 that the vector potential inside a solenoid with n turns per unit length and of radius R is

$$\mathbf{A}=\begin{cases}\frac{\mu_0 nIr}{2}\hat{\boldsymbol{\phi}} & r<R\\ \frac{\mu_0 nIR^2}{2r}\hat{\boldsymbol{\phi}} & r>R\end{cases} \tag{18}$$

Again, we seek components of **W** that satisfy the equation $\mathbf{A}=\nabla\times\mathbf{W}$. Outside the solenoid, we have, using the equations for the curl in cylindrical coordinates:

$$\frac{1}{r}\frac{\partial W_z}{\partial\phi}-\frac{\partial W_\phi}{\partial z} = 0 \tag{19}$$

$$\frac{\partial W_r}{\partial z}-\frac{\partial W_z}{\partial r} = \frac{\mu_0 nIR^2}{2r} \tag{20}$$

$$\frac{1}{r}\left[\frac{\partial\left(rW_\phi\right)}{\partial r}-\frac{\partial W_r}{\partial\phi}\right] = 0 \tag{21}$$

We can try the solution

$$W_r = \frac{\mu_0 nIR^2 z}{2r} \tag{22}$$

$$W_\phi = 0 \tag{23}$$

$$W_z = 0 \tag{24}$$

This satisfies all three curl equations, and also satisfies

$$\nabla\cdot\mathbf{W} = \frac{1}{r}\frac{\partial\left(rW_r\right)}{\partial r}+0+0 \tag{25}$$

$$= 0 \tag{26}$$

Inside the solenoid, we have

$$\frac{1}{r}\frac{\partial W_z}{\partial\phi}-\frac{\partial W_\phi}{\partial z} = 0 \tag{27}$$

$$\frac{\partial W_r}{\partial z}-\frac{\partial W_z}{\partial r} = \frac{\mu_0 nIr}{2} \tag{28}$$

$$\frac{1}{r}\left[\frac{\partial\left(rW_\phi\right)}{\partial r}-\frac{\partial W_r}{\partial\phi}\right] = 0 \tag{29}$$

If we try a similar solution, we have

$$W_r = \frac{\mu_0 n I r z}{2} \tag{30}$$
$$W_\phi = 0 \tag{31}$$
$$W_z = 0 \tag{32}$$

This satisfies all of the curl equations but unfortunately the divergence isn't zero:

$$\nabla\cdot\mathbf{W} = \frac{1}{r}\frac{\partial(rW_r)}{\partial r}+0+0 \tag{33}$$
$$= \mu_0 n I z \tag{34}$$

We can fix this by adding in a z component, so we get

$$W_r = \frac{\mu_0 n I r z}{2} \tag{35}$$
$$W_\phi = 0 \tag{36}$$
$$W_z = -\frac{\mu_0 n I z^2}{2} \tag{37}$$

This doesn't affect the curl, and makes the divergence zero. Again, this solution is unlikely to be unique.