MAGNETIC DIPOLE FIELD OF A FINITE SOLENOID

For our last post on magnetostatics, we’ll consider a finite solenoid of radius $R$ and length $L$, with a surface charge density $\sigma$ rotating at angular speed $\omega$. We know that the field outside an infinite solenoid is zero, but what about a finite solenoid? For points far from the axis, we can use a dipole approximation.

We align the axis along the $z$ axis, and consider it to be a stack of individual current loops, each with its own dipole moment. The moment of a current loop is

$$\mathbf{m} = \pi IR^2 \hat{z}$$  \hspace{1cm} (1)$$

In terms of the parameters of the problem, each loop has a thickness of $dz$ and thus carries a current of $I = \sigma R \omega dz$. The contribution of the loop at coordinate $z$ is therefore

$$d\mathbf{m} = \pi \omega \sigma R^3 dz \hat{z}$$ \hspace{1cm} (2)$$

The field of the dipole from this current loop is

$$d\mathbf{B}_{dip} = \frac{\mu_0}{4\pi r^3} [3 (d\mathbf{m} \cdot \hat{r}) \hat{r} - d\mathbf{m}]$$  \hspace{1cm} (3)$$

We now need to consider carefully what is meant by $\hat{r}$. We’ll take the observation point to be a distance $s$ along a line perpendicular to the axis and intersecting the axis at its midpoint. We’ll define $s$ to be on the $\hat{x}$ axis. Then for a given current loop at coordinate $z$, the vector $\mathbf{r}$ points from the centre of this loop to $s$. Therefore
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\[ r = \sqrt{s^2 + z^2} \]  
\[ \cos \theta = -\frac{z}{r} \]  
\[ \sin \theta = \frac{s}{r} \]  
\[ d\mathbf{m} \cdot \hat{r} = \pi \omega \sigma R^3 \cos \theta \, dz \]  
\[ = -\pi \omega \sigma R^3 \frac{z}{r} \, dz \]

where \( \theta \) is, as usual, the angle between \( \hat{r} \) and \( \hat{z} \). Therefore

\[ \hat{r} = \cos \theta \hat{z} + \sin \theta \hat{x} \]  
\[ = -\frac{z}{r} \hat{z} + \frac{s}{r} \hat{x} \]

The total dipole field of the solenoid is therefore

\[ \mathbf{B}_{\text{dip}} = \frac{\mu_0 \omega \sigma R^3}{4\pi} \int_{-L/2}^{L/2} \left[ \left( \frac{3z^2}{(s^2 + z^2)^{5/2}} - \frac{1}{(s^2 + z^2)^{3/2}} \right) \hat{z} + \frac{sz}{(s^2 + z^2)^{5/2}} \hat{x} \right] \, dz \]

\[ = -\frac{\mu_0 \omega \sigma R^3 L}{4 \left( s^2 + \left( \frac{L}{2} \right)^2 \right)^{3/2}} \hat{z} + 0 \hat{x} \]

\[ = -\frac{\mu_0 \omega \sigma R^3 L}{4 \left( s^2 + \left( \frac{L}{2} \right)^2 \right)^{3/2}} \hat{z} \]

Note that as \( L \to \infty \), the field does tend to zero as \( \frac{1}{L^2} \) which is the correct value for an infinite solenoid.