MUTUAL INDUCTANCE: CALCULATION FROM THE INTEGRAL

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The general formula (known as the Neumann formula) for mutual inductance is

$$M = \frac{\mu_0}{4\pi} \int \int \frac{d\ell_1 \cdot d\ell_2}{|\mathbf{r}_2 - \mathbf{r}_1|}$$

(1)

Although this formula isn’t used much for direct calculation of $M$, we can revisit our initial mutual inductance problem, in which we have two parallel, circular loops sharing a common axis and separated by a distance $z$. The upper loop has radius $a$ and the lower loop $b$. The integral is easiest if we use cylindrical coordinates. Both the line increments are in the $\phi$ direction with the upper increment having magnitude $d\ell_1 = a d\phi_a$ and the lower one $d\ell_2 = b d\phi_b$. The angle between the two increments is $\phi_a - \phi_b$ and both angles are integrated over the range $[0 \ldots 2\pi]$. Therefore

$$d\ell_1 \cdot d\ell_2 = ab \cos(\phi_a - \phi_b) d\phi_a d\phi_b$$

(2)

For the denominator, we have

$$|\mathbf{r}_2 - \mathbf{r}_1| = \left[ (a \cos \phi_a - b \cos \phi_b)^2 + (a \sin \phi_a - b \sin \phi_b)^2 + z^2 \right]^{1/2}$$

(3)

$$= \left[ z^2 + a^2 + b^2 - 2ab \cos(\phi_a - \phi_b) \right]^{1/2}$$

(4)

If we define

$$\beta \equiv \frac{ab}{z^2 + a^2 + b^2}$$

(5)

we get

$$\frac{1}{|\mathbf{r}_2 - \mathbf{r}_1|} = \sqrt{\frac{\beta}{ab}} \frac{1}{\sqrt{1 - 2\beta \cos(\phi_a - \phi_b)}}$$

(6)

We need to do the integral
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\[ M = \frac{\mu_0 \sqrt{ab} \beta}{4\pi} \int_0^{2\pi} \int_0^{2\pi} \frac{\cos(\phi_a - \phi_b)}{\sqrt{1 - 2\beta \cos(\phi_a - \phi_b)}} \, d\phi_a d\phi_b \]  

(7)

This has no closed form solution, but we can expand the denominator in a Taylor series to get (defining \( c \equiv \cos(\phi_a - \phi_b) \)):

\[ \frac{1}{\sqrt{1 - 2\beta \cos(\phi_a - \phi_b)}} = \frac{1}{\sqrt{1 - 2\beta c}} = 1 + \beta c + \frac{3}{2} (\beta c)^2 + \frac{5}{2} (\beta c)^3 + \ldots \]  

(8)

(9)

The integral is then

\[ M = \frac{\mu_0 \sqrt{ab} \beta}{4\pi} \int_0^{2\pi} \int_0^{2\pi} \left[ c + \beta c^2 + \frac{3}{2} \beta^2 c^3 + \frac{5}{2} \beta^3 c^4 + \ldots \right] \, d\phi_a d\phi_b \]  

(10)

(11)

(12)

If the upper loop is much smaller than the lower one and the distance between them (\( a \ll b \) and \( a \ll z \)), then to first order in \( a \), we get \( \beta \approx ab/(b^2 + z^2) \), \( \beta^2 \approx 0 \) and

\[ M \approx \frac{\mu_0}{2} \frac{\pi a^2 b^2}{(b^2 + z^2)^{3/2}} \]  

(13)

which agrees with the earlier result.