DUALITY TRANSFORMATION FOR MAGNETIC AND ELECTRIC CHARGE

Continue our excursion in the fantasy world where magnetic monopoles exist, we can explore the **duality transformations**, which describe a rotation in an E-B space, as follows:

\[
\begin{align*}
E' &= E \cos \alpha + cB \sin \alpha \\
\alpha B' &= cB \cos \alpha - E \sin \alpha \\
cq_e' &= cq_e \cos \alpha + q_m \sin \alpha \\
q_m' &= q_m \cos \alpha - cq_e \sin \alpha
\end{align*}
\]

where \( \alpha \) is an arbitrary angle and \( c = 1/\sqrt{\mu_0 \varepsilon_0} \). Currents and charge densities transform the same way as the corresponding point charges.

It’s a straightforward, though tedious, exercise to verify that Maxwell’s equations (including magnetic charge) are invariant under these transformations. The two equations that are different are:

\[
\begin{align*}
\nabla \cdot B &= \mu_0 \rho_m \\
\nabla \times E &= -\mu_0 J_m - \frac{\partial B}{\partial t}
\end{align*}
\]

A few sample calculations should show how it goes. We’ll do one involving a divergence:

\[
\begin{align*}
c \nabla \cdot B' &= c \nabla \cdot B \cos \alpha - \nabla \cdot E \sin \alpha \\
&= c\mu_0 \rho_m \cos \alpha - \frac{\rho_e}{\varepsilon_0} \sin \alpha \\
&= c\mu_0 (\rho_m \cos \alpha - c\rho_e \sin \alpha) \\
&= c\mu_0 \rho'_m
\end{align*}
\]

And another involving a curl:
The other two equations transform similarly. If we take the force law for electric and magnetic charges to be

$$F = q_e (E + v \times B) + q_m \left( B - \frac{1}{c^2} v \times E \right)$$

(15)

then the transformed law is

$$F' = q'_e (E' + v \times B') + q'_m \left( B' - \frac{1}{c^2} v \times E' \right)$$

(16)

$$= \left( q_e \cos \alpha + \frac{q_m}{c} \sin \alpha \right) \times$$

$$\left[ E \cos \alpha + cB \sin \alpha + v \times \left( B \cos \alpha - \frac{1}{c} E \sin \alpha \right) \right] +$$

(17)

$$\left( q_m \cos \alpha - cq_e \sin \alpha \right) \times$$

$$\left[ B \cos \alpha - \frac{1}{c} E \sin \alpha - \frac{1}{c^2} v \times (E \cos \alpha + cB \sin \alpha) \right]$$

(18)

$$= q_e (E + v \times B) + q_m \left( B - \frac{1}{c^2} v \times E \right)$$

(19)

$$= F$$

(20)

To get the penultimate line it is just a matter of multiplying out the first expression and using \( \cos^2 \alpha + \sin^2 \alpha = 1 \), and cancelling off terms.