ANGULAR MOMENTUM IN ELECTROMAGNETIC FIELDS

The momentum density of an electromagnetic field is given by

\[ p_{em} = \epsilon_0 \mu_0 S = \epsilon_0 \mathbf{E} \times \mathbf{B} \]  

(1)

If we have linear momentum, then we automatically have angular momentum with respect to some origin by using the classical definition of angular momentum \( \mathbf{L} = \mathbf{r} \times \mathbf{p} \). We can define the angular momentum density of an electromagnetic field by

\[ \mathcal{L}_{em} \equiv \mathbf{r} \times p_{em} = \epsilon_0 \mathbf{r} \times (\mathbf{E} \times \mathbf{B}) \]  

(2)

Just as with linear momentum, even static fields can have angular momentum. As an example, suppose we have a long solenoid with \( n \) turns per unit length carrying current \( \mathcal{I}_0 \) and a radius \( R \), with its axis along the \( z \) axis. The magnetic field inside the solenoid is

\[ \mathbf{B}_0 = \mu_0 n \mathbf{I} \hat{z} \]  

(3)

The field is zero outside the solenoid.

Now suppose we add two other cylinders (not solenoids), both coaxial with the solenoid. One cylinder has radius \( a < R \) (so it lies inside the solenoid) and carries surface charge \( +Q \); the other cylinder has radius \( b > R \) (outside the solenoid) and carries charge \( -Q \). Both cylinders have length \( \ell \). From Gauss’s law, the electric field between these two cylinders is, for \( a < r < b \)

\[ \mathbf{E}_0 = \frac{Q}{2 \pi \epsilon_0 \ell} \frac{\hat{r}}{r} \]  

(4)

That is, the field points radially outward from the axis. The electric field is zero for \( r < a \) and \( r > b \). (We’re neglecting end effects, so we’re assuming that \( \ell \gg b > a \).)

The linear momentum density is non-zero in the region \( a < r < R \) (where both fields are non-zero) and we have
\[ p_{em} = -\mu_0 nIQ \frac{\hat{\phi}}{2\pi \ell} \frac{r}{r} \]  

so the angular momentum density is

\[ L_{em} = r \times p_{em} \]  

\[ = -\mu_0 nIQ \frac{\hat{z}}{2\pi \ell} \]  

Conveniently, this is constant so the total angular momentum is just the density times the volume of the cylindrical tube in the region \( a < r < R \)

\[ L_{em} = -\ell \pi \left( R^2 - a^2 \right) \frac{\mu_0 nIQ}{2\pi \ell} \hat{z} \]  

\[ = -\left( R^2 - a^2 \right) \frac{\mu_0 nIQ}{2} \hat{z} \]  

Now suppose we (quasistatically) discharge the two cylinders by connecting a resistor \( \mathcal{R} \) between them. We’d like to show that the angular momentum gets transferred from the fields to the physical devices in the problem. The two cylinders are effectively a capacitor with some capacitance \( C \), so we know that the current in the resistor will decay exponentially

\[ I(t) = \frac{V_0}{\mathcal{R}} e^{-t/\mathcal{R}C} \]  

where \( V_0 \) is the potential difference between the cylinders at \( t = 0 \). The force \( d\mathbf{F} \) on a segment of the resistor of length \( dr \) is

\[ d\mathbf{F} = I(t) dr \hat{r} \times \mathbf{B}_0 \]  

\[ = -I(t) dr B_0 \hat{\phi} \]  

so the torque on this segment is

\[ d\mathbf{N} = r \times d\mathbf{F} \]  

\[ = -I(t) B_0 r dr \hat{z} \]  

The total torque on the resistor at time \( t \) is
\[ \mathbf{N}(t) = -I(t) B_0 \hat{z} \int_a^R r \, dr \]  
\[ = -\frac{1}{2} I(t) B_0 (R^2 - a^2) \hat{z} \] (15)

The angular impulse is the integral of torque over time, so we get

\[ I = \int_0^\infty \mathbf{N}(t) \, dt \] (17)
\[ = -\frac{1}{2} B_0 (R^2 - a^2) \hat{z} \int_0^\infty I(t) \, dt \] (18)
\[ = -\frac{1}{2} B_0 (R^2 - a^2) \hat{z} \int_0^\infty \frac{V_0}{RC} e^{-t/RC} \, dt \] (19)
\[ = -\frac{1}{2} B_0 (R^2 - a^2) CV_0 \hat{z} \] (20)
\[ = -\frac{1}{2} \mu_0 I (R^2 - a^2) Q \hat{z} \] (21)

where we used the relation between capacitance, charge and voltage \( Q = CV \). We see that this agrees with 9, so all the angular momentum in the fields is transferred to the resistor as the electric field is reduced to zero.