WAVES IN A VISCOUS FLUID

Another example of a transverse wave is that of a string embedded in a viscous fluid which adds a drag force on the string. The drag force is proportional to the string’s transverse speed, so we need to insert a term

$$\Delta F_{\text{drag}} = -\gamma \frac{\partial f}{\partial t} \Delta z$$  \hspace{1cm} (1)

where $\gamma$ is a constant determined by the viscosity of the fluid, into our derivation of the original wave equation. This gives us a modified wave equation:

$$T \frac{\partial^2 f}{\partial z^2} - \gamma \frac{\partial f}{\partial t} = \mu \frac{\partial^2 f}{\partial t^2}$$  \hspace{1cm} (2)

where $T$ is the tension in the string and $\mu$ is the mass per unit length of the string.

If we assume that string’s frequency $\omega$ is constant, then we can write

$$f(z,t) = e^{-i\omega t} F(z)$$  \hspace{1cm} (3)

and the wave equation reduces to an ODE:

$$TF'' + i\omega \gamma F = -\mu \omega^2 F$$ \hspace{1cm} (4)

$$F'' = -\frac{i\omega \gamma + \mu \omega^2}{T} F \equiv \alpha^2 F$$ \hspace{1cm} (5)

The general solution is

$$F(z) = Ae^{\alpha z} + Be^{-\alpha z}$$ \hspace{1cm} (6)

However, $\alpha$ is a complex number, so $F$ contains both oscillatory and exponential parts.

Using Maple, we can find the $\alpha$:
\[ \alpha = \frac{1}{\sqrt{2T}} \left[ \sqrt{\omega \sqrt{\omega^2 \mu^2 + \gamma^2} - \mu \omega} - i \sqrt{\omega \sqrt{\omega^2 \mu^2 + \gamma^2} + \mu \omega} \right] \tag{7} \]

To keep \( F(z) \) finite for large \( z \) (we’re assuming the string extends from \( z = 0 \) to \( z = +\infty \)), we must have \( A = 0 \) and therefore

\[ F(z) = A_T \exp \left\{ -\frac{z}{\sqrt{2T}} \left[ \sqrt{\omega \sqrt{\omega^2 \mu^2 + \gamma^2} - \mu \omega} - i \sqrt{\omega \sqrt{\omega^2 \mu^2 + \gamma^2} + \mu \omega} \right] \right\} \tag{8} \]

where \( A_T \) is the (complex) amplitude of the wave.

We can think of the imaginary part of the exponential as the wave number \( k_2 \) and the real part as a spatial decay factor \( \lambda \). That is

\[ k_2 \equiv \frac{1}{\sqrt{2T}} \sqrt{\omega \sqrt{\omega^2 \mu^2 + \gamma^2} + \mu \omega} \tag{9} \]

\[ \lambda \equiv \frac{1}{\sqrt{2T}} \sqrt{\omega \sqrt{\omega^2 \mu^2 + \gamma^2} - \mu \omega} \tag{10} \]

\[ \alpha = \lambda - ik_2 \tag{11} \]

\[ F(z) = A_T e^{-\lambda z + ik_2 z} \tag{12} \]

Because of the negative real part in the exponent, the wave is attenuated (its amplitude falls off with distance) with a characteristic penetration distance \( d \) of

\[ d = \frac{1}{\lambda} = \frac{\sqrt{2T}}{\sqrt{\omega \sqrt{\omega^2 \mu^2 + \gamma^2} - \mu \omega}} \tag{13} \]

This is the distance at which the amplitude falls to \( 1/e \) of its value at \( z = 0 \).

Finally, we can consider the case of two strings joined by a massless knot at \( z = 0 \), with the string on the right embedded in the viscous fluid. If the wave is sinusoidal, then we have

\[ f(z, t) = \begin{cases} A_I e^{i(k_1 z - \omega t)} + A_R e^{i(-k_1 z - \omega t)} & z < 0 \\ e^{-i\omega t} F(z) & z > 0 \end{cases} \tag{14} \]

where \( F(z) \) is given by \( 8 \).

The boundary conditions require the wave function and its derivative to be continuous at \( z = 0 \), so we get
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\[ A_I + A_R = A_T \]  \hspace{1cm} (15)

\[ ik_1 (A_I - A_R) = -\alpha A_T = -\alpha (A_I + A_R) \]  \hspace{1cm} (16)

Solving for \( A_R \) we get

\[ A_R = \frac{\alpha + ik_1 A_I}{\alpha - ik_1} \]  \hspace{1cm} (17)

\[ = \frac{\lambda + i (k_1 - k_2)}{\lambda - i (k_1 + k_2)} A_I \]  \hspace{1cm} (18)

If \( A_I \) is real (that is, the incident wave has zero phase: \( \delta_I = 0 \)) then the magnitude of the amplitude of the reflected wave is

\[ |A_R| = \sqrt{\frac{\lambda^2 + (k_1 - k_2)^2}{\lambda^2 + (k_1 + k_2)^2} A_I} \]  \hspace{1cm} (19)

We can convert (18) by multiplying top and bottom by the complex conjugate of the denominator:

\[ A_R = -\frac{(\lambda + i (k_1 - k_2)) (\lambda + i (k_1 + k_2))}{\lambda^2 + (k_1 + k_2)^2} A_I \]  \hspace{1cm} (20)

\[ = -\frac{\lambda^2 + k_2^2 - k_1^2 + 2i \lambda k_1}{\lambda^2 + (k_1 + k_2)^2} A_I \]  \hspace{1cm} (21)

The phase of the reflected wave is

\[ \tan \delta_R = \frac{\Im (A_R)}{\Re (A_R)} \]  \hspace{1cm} (22)

\[ = \frac{2 \lambda k_1}{\lambda^2 + k_2^2 - k_1^2} \]  \hspace{1cm} (23)

If \( \gamma = 0 \) (so there is no fluid surrounding string 2), then \( \lambda = 0 \) and

\[ A_R = -\frac{k_2^2 - k_1^2}{(k_1 + k_2)^2} A_I \]  \hspace{1cm} (24)

\[ = \frac{k_1 - k_2}{k_2 + k_1} A_I \]  \hspace{1cm} (25)

\[ \tan \delta_R = 0 \]  \hspace{1cm} (26)

which agrees with Griffiths equation 9.29 for sinusoidal waves.