A wave guide is a hollow tube which allows electromagnetic waves to travel down it. Wave guides are usually made of conductors, so we’ll assume that they are made of a perfect conductor so that \( E = 0 \) everywhere inside the conducting boundary.

The boundary conditions implied by Maxwell’s equations are

\[
\begin{align*}
\epsilon_1 E_1^\perp - \epsilon_2 E_2^\perp &= \sigma_f \\
B_1^\perp - B_2^\perp &= 0 \\
E_1^\parallel - E_2^\parallel &= 0 \\
\frac{1}{\mu_1} B_1^\parallel - \frac{1}{\mu_2} B_2^\parallel &= K f \hat{n}
\end{align*}
\]

In particular, \( E_1^\parallel = 0 \) tells us that the parallel component of \( E \) is zero at the boundary of the wave guide. As usual, we’ll take the \( z \) axis to be parallel to wave guide’s axis, so the waves have the form

\[
\begin{align*}
\hat{E} &= \hat{E}_0(x, y) e^{i(kz-\omega t)} \\
\hat{B} &= \hat{B}_0(x, y) e^{i(kz-\omega t)}
\end{align*}
\]

Maxwell’s equations inside the guide (assumed to be a vacuum) are

\[
\begin{align*}
\nabla \cdot E &= 0 \\
\nabla \cdot B &= 0 \\
\nabla \times E &= -\frac{\partial B}{\partial t} \\
\nabla \times B &= \frac{1}{c^2} \frac{\partial E}{\partial t}
\end{align*}
\]

Previously, we applied the divergence equations to show that the waves were transverse (no \( z \) component), but that relied on the waves being unbounded plane waves, which is not the case here. It turns out that waves in...
WAVE GUIDES: DERIVATION OF THE WAVE EQUATION

a wave guide are not transverse in general, in that at least one of \( \mathbf{E} \) and \( \mathbf{B} \) must have a longitudinal component. We therefore write

\[
\tilde{\mathbf{E}}_0(x, y) = E_x \hat{x} + E_y \hat{y} + E_z \hat{z}
\]  
(11)

\[
\tilde{\mathbf{B}}_0(x, y) = B_x \hat{x} + B_y \hat{y} + B_z \hat{z}
\]  
(12)

where all the components on the RHS depend on \( x \) and \( y \), and may be complex functions. Putting these together with 5 and 6 into 9, we get (remembering that the components do not depend on \( z \)):

\[
\nabla \times \mathbf{E} = (\partial_y E_z - ik E_y) \hat{x} + (-\partial_x E_z + ik E_x) \hat{y} + (\partial_x E_y - \partial_y E_x) \hat{z}
\]  
(13)

\[
\nabla \times \mathbf{B} = -\frac{\partial \mathbf{B}}{\partial t}
\]  
(14)

\[
\nabla \times \mathbf{B} = i\omega B_x \hat{x} + i\omega B_y \hat{y} + i\omega B_z \hat{z}
\]  
(15)

Equating components, we get

\[
\partial_y E_z - ik E_y = i\omega B_x
\]  
(16)

\[
-\partial_x E_z + ik E_x = i\omega B_y
\]  
(17)

\[
\partial_x E_y - \partial_y E_x = i\omega B_z
\]  
(18)

We can apply exactly the same procedure to 10 to get the analogous equations

\[
\partial_y B_z - ik B_y = -i\frac{\omega}{c^2} E_x
\]  
(19)

\[
-\partial_x B_z + ik B_x = -i\frac{\omega}{c^2} E_y
\]  
(20)

\[
\partial_x B_y - \partial_y B_x = -i\frac{\omega}{c^2} E_z
\]  
(21)

We can solve these 6 equations to get the \( x \) and \( y \) components in terms of the \( z \) components. For example, multiplying 17 through by \( k \) and 19 through by \( \omega \) and adding, we get

\[
-k \partial_x E_z + ik^2 E_x - i\frac{\omega^2}{c^2} E_x = \omega \partial_y B_z
\]  
(22)

\[
E_x = \frac{1}{i(k^2 - \frac{\omega^2}{c^2})}(k \partial_x E_z + \omega \partial_y B_z)
\]  
(23)

\[
= \frac{i}{\omega^2/c^2 - k^2}(k \partial_x E_z + \omega \partial_y B_z)
\]  
(24)
Similarly, we can get the other 3 equations. Multiply (16) by $k$ and (20) by $\omega$ and subtract to get

$$E_y = \frac{i}{\omega^2/c^2 - k^2} (k\partial_y E_z - \omega \partial_x B_z)$$  \hspace{1cm} (25)

Multiply (16) by $\omega/c^2$ and (20) by $k$ and subtract to get

$$B_x = \frac{i}{\omega^2/c^2 - k^2} \left( k\partial_x B_z - \frac{\omega}{c^2} \partial_y E_z \right)$$ \hspace{1cm} (26)

Multiply (17) by $\omega/c^2$ and (19) by $k$ and add to get

$$B_y = \frac{i}{\omega^2/c^2 - k^2} \left( k\partial_y B_z + \frac{\omega}{c^2} \partial_x E_z \right)$$ \hspace{1cm} (27)

To get the wave equations we can apply (7) and (8):

$$\nabla \cdot E = \frac{i}{\omega^2/c^2 - k^2} \left[ (k\partial_{xx} E_z + \omega \partial_{yy} B_z) + (k\partial_{yy} E_z - \omega \partial_{xx} B_z) \right] + i k E_z$$

$$= \frac{i}{\omega^2/c^2 - k^2} \left[ k\partial_{xx} E_z + k\partial_{yy} E_z \right] + i k E_z$$ \hspace{1cm} (28)

$$= 0$$ \hspace{1cm} (29)

The wave equation for $E_z$ is thus

$$[\partial_{xx} + \partial_{yy} + \omega^2/c^2 - k^2] E_z = 0$$ \hspace{1cm} (31)

Exactly the same procedure applied to $\nabla \cdot B = 0$ gives

$$[\partial_{xx} + \partial_{yy} + \omega^2/c^2 - k^2] B_z = 0$$ \hspace{1cm} (32)

Solving these two equations subject to the boundary conditions will give us all 3 components of each field.

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