Although purely transverse electromagnetic (TEM) waves can’t exist within a hollow wave guide, it is possible to have TEM waves in a coaxial transmission line. To see this, we start with Maxwell’s equations in the form

$$\begin{align*}
\partial_y E_z - ik E_y &= i \omega B_x \\
-\partial_x E_z + ik E_x &= i \omega B_y \\
\partial_x E_y - \partial_y E_x &= i \omega B_z \\
\partial_y B_z - ik B_y &= -i \frac{\omega}{c^2} E_x \\
-\partial_x B_z + ik B_x &= -i \frac{\omega}{c^2} E_y \\
\partial_x B_y - \partial_y B_x &= -i \frac{\omega}{c^2} E_z
\end{align*}$$

By setting $B_z = E_z = 0$ we see from (1) and (5) that

$$B_x = - \frac{k}{\omega} E_y$$

$$-\frac{k^2}{\omega} E_y = - \frac{\omega}{c^2} E_y$$

$$k = \frac{\omega}{c}$$

Substituting this into (1) and (2) we get

$$E_y = -c B_x$$

$$E_x = c B_y$$

from which we see that

$$\mathbf{E} \cdot \mathbf{B} = [c B_y, -c B_x, 0] \cdot [B_x, B_y, 0] = 0$$
so the electric and magnetic fields are perpendicular to each other.

From [3] and the other two Maxwell equations for vacuum: $\nabla \cdot \mathbf{E} = 0$ and $\nabla \cdot \mathbf{B} = 0$ we get

\[
\begin{align*}
\partial_x E_y - \partial_y E_x &= 0 \quad (13) \\
\partial_x B_y - \partial_y B_x &= 0 \quad (14) \\
\partial_x E_x + \partial_y E_y &= 0 \quad (15) \\
\partial_x B_x + \partial_y B_y &= 0 \quad (16)
\end{align*}
\]

Since the components of $\mathbf{E}$ and $\mathbf{B}$ don’t depend on $z$ [remember the $z$ dependence is contained in the complex exponential in the form $\tilde{\mathbf{E}}(x, y, z, t) = \tilde{\mathbf{E}}_0(x, y) e^{i(kz-\omega t)}$ and $\tilde{\mathbf{B}}(x, y, z, t) = \tilde{\mathbf{B}}_0(x, y) e^{i(kz-\omega t)}$] the first equation is equivalent to saying $\nabla \times \tilde{\mathbf{E}}_0 = 0$ and the second to $\nabla \times \tilde{\mathbf{B}}_0 = 0$. Together with $\nabla \cdot \mathbf{E} = 0$ and $\nabla \cdot \mathbf{B} = 0$, these are Maxwell’s equations for static fields $[\frac{\partial \mathbf{E}}{\partial t} = \frac{\partial \mathbf{B}}{\partial t} = 0]$ in empty space (no free charge or current). In a cylindrical coaxial cable with an inner cylinder of radius $a$ and an outer cylinder of radius $b$, the magnetic field is, for $a < r < b$

\[
\mathbf{B}_0 = \frac{\mu_0 I}{2\pi r} \hat{\phi}
\]

where $I$ is the steady current in the inner cylinder. The electric field due to an infinite line of charge is

\[
\mathbf{E}_0 = \frac{\lambda}{2\pi \epsilon_0 r} \hat{r}
\]

where $\lambda$ is the linear charge density. These are formal solutions for the case of cylindrical symmetry; the important thing is that the fields have the forms

\[
\begin{align*}
\mathbf{B}_0 &= \frac{A}{cr} \hat{\phi} \quad (19) \\
\mathbf{E}_0 &= \frac{A}{r} \hat{r} \quad (20)
\end{align*}
\]

for some constant $A$.

Plugging these into the full formulas for $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{B}}$ and taking the real part, we get
COAXIAL WAVE GUIDES: TEM MODE

\[ E = \frac{A \cos (kz - \omega t)}{r} \hat{r} \]  \hspace{1cm} (21)

\[ B = \frac{A \cos (kz - \omega t)}{cr} \hat{\phi} \]  \hspace{1cm} (22)

These equations satisfy Maxwell’s equations:

\[ \nabla \cdot E = \frac{1}{r} \frac{\partial}{\partial r} (rE_r) = 0 \]  \hspace{1cm} (23)

\[ \nabla \cdot B = \frac{1}{r} \frac{\partial B_\phi}{\partial \phi} = 0 \]  \hspace{1cm} (24)

\[ \nabla \times E = \frac{\partial E_r}{\partial z} \hat{\phi} \]  \hspace{1cm} (25)

\[ = -kA \sin (kz - \omega t) \hat{r} \]  \hspace{1cm} (26)

\[ = -\frac{\omega A \sin (kz - \omega t)}{c} r \hat{\phi} \]  \hspace{1cm} (27)

\[ = -\frac{\partial B}{\partial t} \]  \hspace{1cm} (28)

\[ \nabla \times B = -\frac{\partial B_\phi}{\partial z} \hat{r} + \frac{1}{r} \frac{\partial}{\partial r} (rB_\phi) \hat{z} \]  \hspace{1cm} (29)

\[ = k \frac{A \sin (kz - \omega t)}{cr} \hat{r} \]  \hspace{1cm} (30)

\[ = \frac{\omega A \sin (kz - \omega t)}{c^2 r} \hat{r} \]  \hspace{1cm} (31)

\[ = \frac{1}{c^2} \frac{\partial E}{\partial t} \]  \hspace{1cm} (32)

The boundary conditions are

\[ B_1^\perp - B_2^\perp = 0 \]  \hspace{1cm} (33)

\[ E_1^\parallel - E_2^\parallel = 0 \]  \hspace{1cm} (34)

and since \( B \) is circumferential, its component normal to the cylinders is zero, and since \( E \) is radial, its parallel component is zero, so the boundary conditions are satisfied.

By comparison with [17] and [18] we can write the analogs for the full fields

\[ B = \frac{\mu_0 I (z,t)}{2\pi r} \hat{\phi} \]  \hspace{1cm} (35)
\[ E = \frac{\lambda(z,t)}{2\pi \varepsilon_0 r^2} \hat{r} \] (36)

From this we can read off the current and charge density on the inner cylinder.

\[ I(z,t) = \frac{2\pi A \cos(kz - \omega t)}{\mu_0 c} \] (37)

\[ \lambda(z,t) = 2\pi \varepsilon_0 A \cos(kz - \omega t) \] (38)