JEFIMENKO’S EQUATION FOR TIME-DEPENDENT ELECTRIC FIELD

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Using the retarded potentials, we can find a time-dependent expression for the electric field \( \mathbf{E} \). The potentials are

\[
V(\mathbf{r}, t) = \frac{1}{4\pi \varepsilon_0} \int \frac{\rho(\mathbf{r}', t)}{d} d^3 \mathbf{r}' \quad (1)
\]

\[
A(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}', t)}{d} d^3 \mathbf{r}' \quad (2)
\]

where

\[
t_r \equiv t - \frac{d}{c} \quad (3)
\]

and

\[
d \equiv |\mathbf{r} - \mathbf{r}'| \quad (4)
\]

\[
d = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2} \quad (5)
\]

\[
\hat{\mathbf{d}} = \frac{\mathbf{r} - \mathbf{r}'}{d} \quad (6)
\]

The expression for \( \mathbf{E} \) is given by

\[
\mathbf{E} = -\nabla V - \frac{\partial A}{\partial t} \quad (7)
\]

The derivatives are complicated by the fact that the integrands in \( 1 \) and \( 2 \) depend on \( \mathbf{r} \) both via \( t_r \) and \( d \). We get (note that \( \nabla \) indicates derivatives with respect to components of \( \mathbf{r} \) only, not \( \mathbf{r}' \)):
\[
\n\nabla V = \frac{1}{4\pi \epsilon_0} \int \nabla \left( \frac{\rho(r', t_r)}{d} \right) d^3 r' \\
= \frac{1}{4\pi \epsilon_0} \int \rho \nabla \left( \frac{1}{d} \right) + \frac{1}{d} \nabla \rho d^3 r'
\]

(8)

(9)

Using the chain rule

\[
\nabla \rho = \frac{\partial \rho}{\partial t_r} \nabla t_r \\
= -\frac{1}{c} \frac{\partial \rho}{\partial t} \nabla d \\
= -\frac{\dot{\rho}}{c} \nabla d
\]

(10)

(11)

(12)

since \( \frac{\partial \rho}{\partial t_r} = \frac{\partial \rho}{\partial t} \) because of \(3\).

By direct calculation we have

\[
\nabla d = \hat{d} \\
\n\nabla \left( \frac{1}{d} \right) = -\frac{\hat{d}}{d^2}
\]

(13)

(14)

Plugging everything into (9) we get

\[
\nabla V (r, t) = -\frac{1}{4\pi \epsilon_0} \int \left( \frac{\rho}{d^2} + \frac{\dot{\rho}}{cd} \right) \hat{d} d^3 r' 
\]

(15)

The second term in (7) is just

\[
\frac{\partial A}{\partial t} = \frac{\mu_0}{4\pi} \int \frac{J(r', t_r)}{d} d^3 r'
\]

(16)

so the time-dependent field is (using \( \mu_0 \epsilon_0 = 1/c^2 \)):

\[
\nabla V (r, t) = \frac{1}{4\pi \epsilon_0} \int \left( \frac{\rho(r', t_r)}{d^2} + \frac{\dot{\rho}(r', t_r)}{cd} \right) \hat{d} d^3 r' - \frac{\mu_0}{4\pi} \frac{J(r', t_r)}{d} d^3 r' \\
= \frac{1}{4\pi \epsilon_0} \int \left( \frac{\rho(r', t_r)}{d^2} \hat{d} + \frac{\dot{\rho}(r', t_r)}{cd} \hat{d} - \frac{J(r', t_r)}{c^2 d} \right) d^3 r'
\]

(17)

(18)

This is Jefimenko’s equation for the electric field. In the static case, all time derivatives are zero and there is no dependence on \( t_r \) so we get

\[
\nabla V (r) = \frac{1}{4\pi \epsilon_0} \int \frac{\rho(r')}{d^2} \hat{d} d^3 r'
\]

(19)
which is just Coulomb’s law from electrostatics. A special case is that of constant current but varying charge. In that case

\[ \rho(r,t) = \dot{\rho}(r,0) t + \rho(r,0) \]  \hspace{1cm} (20)

where

\[ \dot{\rho}(r,0) = -\nabla \cdot \mathbf{J}(r) \]  \hspace{1cm} (21)

In this case, the integrand of 18 is

\[ \frac{\rho(r',t_r)}{d^2} \hat{\mathbf{d}} + \frac{\dot{\rho}(r',t_r)}{c d} \frac{\mathbf{d}}{c^2 d} = \frac{\dot{\rho}(r',0) (t - \frac{d}{c}) + \rho(r',0)}{d^2} \hat{\mathbf{d}} + \frac{\dot{\rho}(r',0)}{c d} \hat{\mathbf{d}} \]  \hspace{1cm} (22)

\[ = \frac{\dot{\rho}(r',0) t + \rho(r',0)}{d^2} \hat{\mathbf{d}} \]  \hspace{1cm} (23)

\[ = \frac{\rho(r',t)}{d} \hat{\mathbf{d}} \]  \hspace{1cm} (24)

so the field is

\[ \mathbf{E}(r,t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(r',t)}{d^2} \hat{d} d^3 r' \]  \hspace{1cm} (25)

That is, Coulomb’s law is valid with the charge density evaluated at the current (non-retarded) time.

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