FIELDS DUE TO A MOVING LINEAR CHARGE

In his example 10.4, Griffiths works out the fields due to a point charge moving with constant velocity $v$. They are

$$E(r,t) = \frac{q}{4\pi\varepsilon_0} \frac{1-v^2/c^2}{(1-v^2 \sin^2 \theta/c^2)^{3/2}} \hat{R}$$

(1)

where

$$R \equiv r - vt$$

(2)

is the vector from the particle’s present (not retarded) position to the observer (assuming the particle passes through the origin at $t = 0$) and $\theta$ is the angle between $R$ and $v$. We can use this formula to rederive the equation for the electric field due to an infinite line charge with linear charge density $\lambda$. From electrostatics, we know the field is given by

$$E = \frac{\lambda}{2\pi\varepsilon_0 z}$$

(3)

where $z$ is the perpendicular distance from the line (wire). Let’s see if we can get the same result using the formula above.

The field due to a small segment of the wire of length $dx$ at position is that due to a point charge $\lambda dx$. For an observation point at $r$, the length of $R$ is

$$R = \sqrt{z^2 + x^2}$$

(4)

and since the velocity is parallel to the wire, we have

$$\sin \theta = \frac{z}{\sqrt{z^2 + x^2}}$$

(5)
Since \( \mathbf{E} \) is parallel to \( \mathbf{R} \), by symmetry the components of \( \mathbf{E} \) parallel to the wire will cancel out, since there will be equal and opposite contributions from points \( \pm x \). The perpendicular component is \( \mathbf{E} \sin \theta \) so the total field is

\[
\mathbf{E} = \frac{\lambda}{4\pi\varepsilon_0} \left(1 - \frac{v^2}{c^2}\right) \int_{-\infty}^{\infty} \frac{\sin \theta dx}{\left(1 - v^2 \sin^2 \theta / c^2\right)^{3/2} (z^2 + x^2)} \hat{s}
\]  

(6)

where the \( s \) direction is radial. We can convert this to an integral over \( \theta \) by noting that

\[
\cos \theta d\theta = -\frac{xz}{(z^2 + x^2)^{3/2}} dx
\]  

(7)

\[
\frac{dx}{z^2 + x^2} = -\frac{\sqrt{z^2 + x^2}}{xz} \left(\frac{xz}{(z^2 + x^2)^{3/2}} dx\right)
\]  

(8)

\[
= -\frac{\sqrt{z^2 + x^2}}{xz} \cos \theta d\theta
\]  

(9)

But

\[
\cos \theta = -\frac{x}{\sqrt{z^2 + x^2}}
\]  

(10)

so

\[
\frac{\lambda}{4\pi\varepsilon_0} \left(1 - \frac{v^2}{c^2}\right) \int_{-\infty}^{\infty} \frac{\sin \theta dx}{\left(1 - v^2 \sin^2 \theta / c^2\right)^{3/2} (z^2 + x^2)} \hat{s} = \frac{dx}{z^2 + x^2} \frac{d\theta}{z}
\]  

(11)

\[
\int_{-\infty}^{\infty} \frac{\sin \theta dx}{\left(1 - v^2 \sin^2 \theta / c^2\right)^{3/2} (z^2 + x^2)} \cdot \frac{\hat{s}}{z}
\]  

(12)

The integral can be evaluated using Maple, and we get

\[
\int_{0}^{\pi} \frac{\sin \theta d\theta}{\left(1 - v^2 \sin^2 \theta / c^2\right)^{3/2}} \hat{s} = \frac{-\cos \theta}{\left(1 - \frac{v^2}{c^2}\right) \sqrt{1 - \frac{v^2}{c^2} \sin^2 \theta}} \hat{s} \bigg|_{0}^{\pi}
\]  

(13)

\[
= \frac{2}{1 - \frac{v^2}{c^2}} \hat{s}
\]  

(14)

so we get back the correct field.
The magnetic field of a point charge is given by Griffiths as

\[ \mathbf{B}(\mathbf{r},t) = \frac{\mathbf{v} \times \mathbf{E}(\mathbf{r},t)}{c^2} \]  
(16)

Since \( \mathbf{v} \) is a constant, the total magnetic field can be found from the same integral as above. Its direction is given by \( \hat{x} \times \hat{s} = \hat{\phi} \) which circles the wire in a direction given by the usual right-hand rule. Since \( \lambda \mathbf{v} = \mathbf{I} \) (the current), we get

\[ \mathbf{B} = \frac{\mathbf{I}}{2\pi \epsilon_0 c^2 z} \hat{\phi} \]  
(17)

\[ = \frac{\mu_0 \mathbf{I}}{2\pi z} \hat{\phi} \]  
(18)

which agrees with the magnetostatic formula using \textit{Ampère’s law}.