FORCE OF POINT CHARGE IN A HYPERBOLIC
TRAJECTORY ON A FIXED POINT CHARGE

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We’ll now return to the point charge $q_2$ moving on a hyperbolic trajectory along the $x$ axis. Its position is given by

$$x(t) = \sqrt{b^2 + c^2 t^2}$$

Suppose there is a second charge $q_1$ fixed at $x = 0$. What force does $q_1$ exert on $q_2$ at time $t$? If we look at the problem from $q_2$’s perspective, it needs to know what influence $q_1$ has on the point $x$ that it ($q_2$, that is) currently occupies. In order for a signal to reach $x$ at time $t$, it had to leave $q_1$ at time $t_r = t - \frac{x}{c}$. But since $q_1$ is fixed at the origin, the force that $q_1$ exerts on any charge $q_2$ at the point $x$ is always just given by Coulomb’s law without any retarded time, that is

$$F_2(t) = \frac{q_1 q_2}{4\pi \varepsilon_0 x^2}$$

$$= \frac{q_1 q_2}{4\pi \varepsilon_0} \frac{1}{b^2 + c^2 t^2}$$

The total impulse delivered to $q_2$ is

$$I_2 = \int_{-\infty}^{\infty} F_2(t) \, dt$$

$$= \frac{q_1 q_2}{4\pi \varepsilon_0 b^2} \int_{-\infty}^{\infty} \frac{1}{1 + \frac{c^2}{b^2} t^2} \, dt$$

$$= \frac{q_1 q_2}{4\pi \varepsilon_0 b c}$$

To calculate the force $q_2$ exerts on $q_1$ we do need the retarded time, since $q_2$ is moving. [Incidentally, it might seem that by considering $q_1$ as at rest and $q_2$ as moving, and treating the two cases differently, we’re violating the principle of relativity, but we’re not. The reason is that $q_2$ is not in an inertial
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frame; the hyperbolic trajectory means that its velocity is never constant, so $q_1$ and $q_2$ are not equivalent.]

The retarded time is calculated from

$$|\mathbf{r} - \mathbf{w}(t_r)| = c(t - t_r)$$  \hspace{0.5cm} (7)

Here $\mathbf{r} = 0$ is the location of $q_1$ and $\mathbf{w}(t_r) = \sqrt{b^2 + c^2t_r^2}\hat{x}$ is the position of $q_2$. We get

$$\sqrt{b^2 + c^2t_r^2} = c(t - t_r)$$  \hspace{0.5cm} (8)

$$t_r = \frac{t}{2} - \frac{b^2}{2c^2t}$$  \hspace{0.5cm} (9)

Note that as $t \to 0$, $t_r \to -\infty$ so $q_2$ is not visible to $q_1$ before $t = 0$, so the retarded potential is zero for $t < 0$. The force for $t > 0$ is therefore (the minus sign indicates the force is to the left if both charges are the same sign)

$$F_1(t) = -\frac{q_1q_2}{4\pi\epsilon_0} \frac{1}{b^2 + c^2\left(t - \frac{b^2}{c^2t}\right)^2}$$  \hspace{0.5cm} (10)

$$= -\frac{q_1q_2}{\pi\epsilon_0} \frac{c^2t^2}{4c^2b^2t^2 + (c^2t^2 - b^2)^2}$$  \hspace{0.5cm} (11)

$$= -\frac{q_1q_2}{\pi\epsilon_0} \frac{c^2t^2}{(c^2t^2 + b^2)^2}$$  \hspace{0.5cm} (12)

The total impulse is found by integrating $F_1$ from $t = 0$ to infinity, since there is no force for $t < 0$.

$$I_1 = -\frac{q_1q_2}{\pi\epsilon_0} \int_0^\infty \frac{c^2t^2}{(c^2t^2 + b^2)^2} dt$$  \hspace{0.5cm} (13)

$$= -\frac{q_1q_2}{4\pi\epsilon_0bc}$$  \hspace{0.5cm} (14)

[The integral can be done by parts, although I used Maple.] Thus the two impulses are equal and opposite.