RADIATION FROM A ROTATING DIPOLE

We can simulate a rotating dipole by superimposing two perpendicular oscillating dipoles. If our rotating dipole is located at the origin and rotates about the \( z \) axis (so the axis of the dipole lies in the \( xy \) plane), then we get

\[
p = p_0 (\cos \omega t \hat{x} + \sin \omega t \hat{y})
\]  

(1)

Since the fields obey the superposition principle (fields from 2 sources just add), we can work through the formulas we found earlier to get the fields and thus the radiated power. To simplify the notation, we’ll use the shorthand

\[
c_\omega \equiv \cos \left( \omega \left( t - \frac{r}{c} \right) \right)
\]  

(2)

\[
c_\theta \equiv \cos \theta
\]  

(3)

\[
c_\phi \equiv \cos \phi
\]  

(4)

with analogous notation for the sines of these quantities.

The fields for a dipole pointing in an arbitrary direction are

\[
E = -\frac{\mu_0 \omega^2}{4\pi r} \cos \left( \omega \left( t - \frac{r}{c} \right) \right) \left( \hat{p}_0 \times \hat{r} \right) \times \hat{r}
\]  

(5)

\[
B = -\frac{\mu_0 \omega^2}{4\pi r c} \cos \left( \omega \left( t - \frac{r}{c} \right) \right) \left( \hat{p}_0 \times \hat{r} \right)
\]  

(6)

Superposing the two perpendicular dipoles we get

\[
E = -\frac{\mu_0 p_0 \omega^2}{4\pi r} [(c_\omega \hat{x} + s_\omega \hat{y}) \times \hat{r}] \times \hat{r}
\]  

(7)

\[
B = -\frac{\mu_0 p_0 \omega^2}{4\pi r c} [(c_\omega \hat{x} + s_\omega \hat{y}) \times \hat{r}]
\]  

(8)

To do the cross products we convert the rectangular unit vectors to spherical unit vectors:
\[ \hat{x} = s_\theta c_\phi \hat{r} + c_\theta c_\phi \hat{\theta} - s_\phi \hat{\phi} \] (9)

\[ \hat{y} = s_\theta s_\phi \hat{r} + c_\theta s_\phi \hat{\theta} + c_\phi \hat{\phi} \] (10)

Then

\[ \hat{x} \times \hat{r} = -c_\theta c_\phi \hat{\phi} - s_\phi \hat{\theta} \] (11)

\[ (\hat{x} \times \hat{r}) \times \hat{r} = -c_\theta s_\phi \hat{\phi} + c_\phi \hat{\theta} \] (12)

\[ \hat{y} \times \hat{r} = -c_\theta s_\phi \hat{\phi} - c_\phi \hat{\theta} \] (13)

\[ (\hat{y} \times \hat{r}) \times \hat{r} = -c_\theta s_\phi \hat{\phi} - c_\phi \hat{\theta} \] (14)

Plugging everything in and collecting terms we get

\[ \mathbf{E} = -\frac{\mu_0 p_0 \omega^2}{4\pi r} \left[ \hat{\theta} c_\theta \left( -c_\omega c_\phi - s_\omega s_\phi \right) + \hat{\phi} \left( c_\omega s_\phi - s_\omega c_\phi \right) \right] \] (15)

\[ \mathbf{B} = -\frac{\mu_0 p_0 \omega^2}{4\pi r c} \left[ -\hat{\theta} \left( c_\omega s_\phi - s_\omega c_\phi \right) + \hat{\phi} \left( -c_\omega c_\phi - s_\omega s_\phi \right) \right] \] (16)

Defining

\[ E_\theta \equiv -c_\theta \left( c_\omega c_\phi + s_\omega s_\phi \right) \] (17)

\[ E_\phi \equiv \left( c_\omega s_\phi - s_\omega c_\phi \right) \] (18)

we can write the fields as

\[ \mathbf{E} = -\frac{\mu_0 p_0 \omega^2}{4\pi r} \left[ \hat{\theta} E_\theta + \hat{\phi} E_\phi \right] \] (19)

\[ \mathbf{B} = -\frac{\mu_0 p_0 \omega^2}{4\pi r c} \left[ -\hat{\theta} E_\phi + \hat{\phi} E_\theta \right] \] (20)

In this form, it’s obvious that \( \mathbf{E} \cdot \mathbf{B} = 0 \), so \( \mathbf{E} \) and \( \mathbf{B} \) are perpendicular.

The Poynting vector is

\[ \mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B} \] (21)

\[ = \frac{\mu_0}{c} \left( \frac{p_0 \omega^2}{4\pi r} \right)^2 \left( E_\theta^2 + E_\phi^2 \right) \hat{r} \] (22)

\[ = \frac{\mu_0}{c} \left( \frac{p_0 \omega^2}{4\pi r} \right)^2 \left( c_\theta^2 \left( c_\omega c_\phi + s_\omega s_\phi \right)^2 + \left( c_\omega s_\phi - s_\omega c_\phi \right)^2 \right) \hat{r} \] (23)
The average energy radiated is the average of $S$ over a single time cycle, so it’s the average over the terms involving $c_\omega$ and $s_\omega$. These are of two types: terms involving $c_\omega^2$ or $s_\omega^2$ and the cross terms involving $s_\omega c_\omega$. The average of $s_\omega c_\omega$ over a cycle is zero and the average of $c_\omega^2$ or $s_\omega^2$ is $\frac{1}{2}$, so the cross terms contribute nothing and we get

$$\langle S \rangle = \frac{\mu_0}{c} \left( \frac{p_0 \omega^2}{4\pi r} \right)^2 \left[ \frac{1}{2} c_\theta^2 (c_\phi^2 + s_\phi^2) + \frac{1}{2} (c_\phi^2 + s_\phi^2) \right]$$

$$= \frac{\mu_0}{2c} \left( \frac{p_0 \omega^2}{4\pi r} \right)^2 \left( 1 + \cos^2 \theta \right)$$

(24)

The average radiated power is maximum in the $\pm z$ directions where $\theta = 0, \pi$ and minimum (though not zero) in the $xy$ plane, where $\theta = \frac{\pi}{2}$. There is no dependence on $\phi$ which is what we’d expect on average since the dipole rotates uniformly through all values of $\phi$. [There is a dependence on $\phi$ within each cycle, since the radiated power in a given azimuthal direction depends on where the dipole is in its rotation.]

The total average radiated power is

$$\langle P \rangle = \frac{\mu_0}{2c} \left( \frac{p_0 \omega^2}{4\pi} \right)^2 \int_0^\pi \int_0^{2\pi} \frac{1 + \cos^2 \theta}{r^2} r^2 \sin \theta d\phi d\theta$$

$$= \frac{\mu_0 p_0^2 \omega^4}{6\pi c}$$

(25)

This is exactly twice the power from a single oscillating dipole. Although power doesn’t ordinarily obey the superposition principle since it depends on the product of $E$ and $B$, it does here because the cross terms in 23 average out to zero over a time cycle, since the two perpendicular dipoles are $\frac{\pi}{2}$ out of phase. If they were exactly in phase, we would replace $s_\omega$ by $c_\omega$ everywhere in the calculation, and then the cross terms wouldn’t average out to zero and the combined power would not be twice the individual power.

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