Radiation reaction with a delta-function external force

Here’s another example of applying an external force to a charge feeling the radiation reaction force. In general, a charge’s acceleration obeys the differential equation

\[ a = \tau \dot{a} + \frac{F}{m} \]  

(1)

where \( F \) is the external force and

\[ \tau \equiv \frac{\mu_0 q^2}{6\pi mc} \]  

(2)

Suppose now that the force is a delta function:

\[ F = k\delta(t) \]  

(3)

for some constant \( k \). In the earlier post, we showed that if \( F \) is finite everywhere, then \( a \) must be continuous everywhere. However, here \( F \) is not finite at \( t = 0 \). As before we start by integrating \[ 1 \] over a small time interval around \( t = 0 \):

\[ \int_{-\epsilon}^{\epsilon} a \, dt = \tau [a(\epsilon) - a(-\epsilon)] + \frac{1}{m} \int_{-\epsilon}^{\epsilon} F \, dt \]  

(4)

Provided that \( a \) is finite everywhere, the integral on the LHS goes to zero as \( \epsilon \to 0 \) so we’re left with
\[
\tau \Delta a = -\frac{1}{m} \int_{-\epsilon}^{\epsilon} F \, dt \tag{5}
\]

\[
= -\frac{k}{m} \int_{-\epsilon}^{\epsilon} \delta(t) \, dt \tag{6}
\]

\[
= -\frac{k}{m} \tag{7}
\]

\[
\Delta a = -\frac{k}{m\tau} \tag{8}
\]

We can repeat the calculations we did earlier to check that energy is conserved here. Since \( F = 0 \) everywhere except \( t = 0 \), the general solution of \( \Delta a \) is

\[
a(t) = \begin{cases} 
a_0 e^{t/\tau} & t < 0 \\ a_1 e^{t/\tau} & t > 0 \end{cases} \tag{9}
\]

If we eliminate the runaway acceleration for \( t > 0 \) by requiring \( a_1 = 0 \) then the condition \( \Delta a \) requires \( a_0 = \frac{k}{m\tau} \), so

\[
a(t) = \begin{cases} 
\frac{k}{m\tau} e^{t/\tau} & t < 0 \\
0 & t > 0 \end{cases} \tag{10}
\]

By requiring \( v = 0 \) at \( t = -\infty \) and that \( v \) is continuous at \( t = 0 \) we get

\[
v(t) = \begin{cases} 
\frac{k}{m} e^{t/\tau} & t < 0 \\
\frac{k}{m} & t > 0 \end{cases} \tag{11}
\]

The work done by the force is

\[
W = \int_{-\infty}^{\infty} F v \, dt \tag{12}
\]

\[
= k \int_{-\infty}^{\infty} \delta(t) v \, dt \tag{13}
\]

\[
= kv(0) \tag{14}
\]

\[
= \frac{k^2}{m} \tag{15}
\]

The energy radiated \( R \) is given by integrating the Larmor formula

\[
P = \frac{\mu_0 q^2 a^2}{6\pi c} = m\tau a^2 \tag{16}
\]
so we get

\[ R = m\tau \int_{-\infty}^{0} \left( \frac{k}{m\tau} \right)^2 e^{2t/\tau} dt \]  \hspace{1cm} (17)

\[ = \frac{k^2}{2m} \]  \hspace{1cm} (18)

The final kinetic energy is

\[ K = \frac{1}{2} m \frac{k^2}{m^2} = \frac{k^2}{2m} \]  \hspace{1cm} (19)

Thus

\[ W = R + K \]  \hspace{1cm} (20)

and energy is conserved.

**Pingbacks**

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