TUNNELLING THROUGH A POTENTIAL BARRIER WITH THE RADIATION REACTION FORCE

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Besides giving rise to runaway acceleration and violation of causality, the radiation reaction force also predicts a classical version of tunnelling through a potential barrier, something you might think is confined to quantum mechanics. Suppose a charged particle travels in along the x axis, starting at \( x = -\infty \) with some initial velocity \( v_i \). Between \( x = 0 \) and \( x = L \) there is a finite potential barrier of height \( U_0 \).

In general, the charge’s acceleration obeys the differential equation

\[
a = \tau \dot{a} + \frac{F}{m}
\]  

(1)

where \( F \) is the external force and

\[
\tau \equiv \frac{\mu_0 q^2}{6\pi mc}
\]  

(2)

Because the derivative of a step function is delta function and the force is the negative gradient of the potential, the force is

\[
F = U_0 (\delta(x) + \delta(x - L))
\]  

(3)

We’ve treated radiation reaction with a delta function force before, but in that case, the force was a delta function in time rather than space. However, we can envision the particle travelling in until it arrives at \( x = 0 \) at which point it feels a delta function force, then travelling along to \( x = L \) where it feels another delta function force. Previously, to solve (1) around the time \( t = 0 \) we integrated the equation over a small interval about \( t = 0 \):

\[
\int_{-\epsilon}^{\epsilon} a \, dt = \tau [a(\epsilon) - a(-\epsilon)] + \frac{1}{m} \int_{-\epsilon}^{\epsilon} F \, dt
\]  

(4)
If \( F \) were a delta function in time, the integral is easily done, as \( \int_{-\epsilon}^{\epsilon} \delta(t) \, dt = 1 \). In our case, with the delta function a function of position, we can consider the position as a function of time and use the chain rule:

\[
\int_{-\epsilon}^{\epsilon} \delta(x(t)) \, dt = \int_{-\epsilon}^{\epsilon} \delta(u) \, du
\]

where \( u \equiv x(t) \) so that \( du = \dot{x} \, dt = v \, dt \) and \( dt = du/v \). Therefore

\[
\int_{-\epsilon}^{\epsilon} \delta(x(t)) \, dt = \int_{-\epsilon}^{\epsilon} \frac{\delta(u)}{v} \, du = \frac{1}{v(u=0)}
\]

The velocity in the last line is evaluated at \( u = 0 \), which corresponds to \( x = 0 \). If we define our origin of time at this point, then this is also the velocity at \( t = 0 \), so we’ve converted the problem into the one we’ve already solved.

In our earlier solution, we saw that the acceleration for a force of \( k\delta(t) \) has a discontinuity at \( t = 0 \), with \( \Delta a = -k/m\tau \). Here, \( k = -U_0/v_0 \), where \( v_0 \) is the velocity at \( t = 0 \), so

\[
\Delta a_0 = +\frac{U_0}{m\tau v_0}
\]

Similarly, at \( x = L \) we can say that the particle reaches this point at \( t = T \) when the force is equal in magnitude but opposite in direction, so

\[
\Delta a_T = -\frac{U_0}{m\tau v_T}
\]

where \( v_T \) is the velocity at \( t = T \). The general solution for the acceleration is therefore

\[
a(t) = \begin{cases} a_0 e^{t/\tau} & t < 0 \\ a_1 e^{t/\tau} & 0 < t < T \\ a_2 e^{t/\tau} & t > T \end{cases}
\]

If we set \( a_2 = 0 \) to prevent runaway acceleration for \( t > T \), then we can apply Eqs. 8 and 9 to determine \( a_1 \) and \( a_0 \). We get at \( t = T \)

\[
a_1 = \frac{U_0}{m\tau v_T} e^{-T/\tau}
\]
Because $a_2 = 0$, there is no further acceleration after $t = T$, so the velocity at this time remains constant for all future times, and we can write it as $v_f$, the final velocity. Therefore

$$a_1 = \frac{U_0}{m\tau v_f} e^{-T/\tau} \quad (12)$$

At $t = 0$ we get

$$a_1 - a_0 = \frac{U_0}{m\tau v_0} \quad (13)$$

$$a_0 = \frac{U_0}{m\tau} \left( \frac{e^{-T/\tau}}{v_f} - \frac{1}{v_0} \right) \quad (14)$$

In summary,

$$a(t) = \begin{cases} \frac{U_0}{m\tau} \left( \frac{e^{-t/\tau}}{v_f} - \frac{1}{v_0} \right) e^{t/\tau} & t < 0 \\ \frac{U_0}{mv_f} e^{-T/\tau} e^{t/\tau} & 0 < t < T \\ 0 & t > T \end{cases} \quad (15)$$

Integrating this to get the velocity, we have

$$v(t) = \begin{cases} \frac{U_0}{m} \left( \frac{e^{-t/\tau}}{v_f} - \frac{1}{v_0} \right) e^{t/\tau} + v_i & t < 0 \\ \frac{U_0}{mv_f} e^{-T/\tau} e^{t/\tau} + v_1 & 0 < t < T \\ v_f & t > T \end{cases} \quad (16)$$

Integrating again, we get the position

$$x(t) = \begin{cases} \frac{U_0\tau}{m} \left( \frac{e^{-t/\tau}}{v_f} - \frac{1}{v_0} \right) e^{t/\tau} + v_i t + x_0 & t < 0 \\ \frac{U_0\tau}{mv_f} e^{-T/\tau} e^{t/\tau} + v_1 t + x_1 & 0 < t < T \\ v_f (t - T) + L & t > T \end{cases} \quad (17)$$

where in the last line, we’ve just imposed the condition that the particle moves at a constant velocity $v_f$ starting from $x = L$ at $t = T$.

We now apply boundary conditions to find the various constants. First, to get $x_0$ we require $x(0) = 0$, which gives

$$x_0 = \frac{U_0\tau}{mv_0 v_f} \left( v_f - v_0 e^{-T/\tau} \right) \quad (18)$$

Also using $t = 0$, we can find $x_1$ from the middle expression for $x(t)$:
\[ x_1 = -\frac{U_0 \tau}{mv_f} e^{-T/\tau} \] (19)

Also from the middle expression for \( x(t) \) we can find \( v_1 \) by requiring \( x = L \) at \( t = T \):

\[ v_1 = \frac{1}{T} \left( L - \frac{U_0 \tau}{mv_f} \left( 1 - e^{-T/\tau} \right) \right) \] (20)

We can now plug this into the middle expression for \( v(t) \), set \( t = 0 \) and require the velocity to be \( v_0 \) to find \( v_0 \):

\[ v_0 = \frac{1}{T} \left( L + \frac{U_0}{mv_f} \left( -\tau + (\tau + T) e^{-T/\tau} \right) \right) \] (21)

Substituting this back into the middle expression for \( v(t) \) at \( t = T \), when the velocity is \( v_f \) we find

\[ L = v_f T - \frac{U_0}{mv_f} \left( T - \tau \left( 1 - e^{-T/\tau} \right) \right) \] (22)

Finally, we can find \( v_i \) by taking the first expression for \( v(t) \), setting \( t = 0 \) and requiring the result to be equal to \( v_0 \). This gives a rather unpleasant expression which can be simplified by collecting terms.
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\[
v_i = \frac{m^2 v_i^4 - U_0 m \left( 1 - e^{-T/\tau} \right) v_f^2 + U_0^2 \left( 1 - e^{-T/\tau} \right)}{mv_f \left( mv_f^2 - U_0 \left( 1 - e^{-T/\tau} \right) \right)} \tag{23}
\]

\[
v_i = v_f - \frac{U_0}{mv_f} \left( \frac{U_0 \left( e^{-T/\tau} - 1 \right)}{mv_f^2 + U_0 \left( e^{-T/\tau} - 1 \right)} \right) \tag{24}
\]

\[
v_i = v_f - \frac{U_0}{mv_f} \left( \frac{mv_f^2 + U_0 \left( e^{-T/\tau} - 1 \right) - mv_f^2 U_0 \left( e^{-T/\tau} - 1 \right) + U_0 \left( e^{-T/\tau} - 1 \right)}{mv_f^2 + U_0 \left( e^{-T/\tau} - 1 \right)} \right) \tag{25}
\]

\[
v_i = v_f - \frac{U_0}{mv_f} \left( \frac{1}{mv_f^2 + U_0 \left( e^{-T/\tau} - 1 \right)} \right) \tag{26}
\]

\[
v_i = v_f - \frac{U_0}{mv_f} \left( 1 - \frac{1}{1 + \frac{U_0}{mv_f} \left( e^{-T/\tau} - 1 \right)} \right) \tag{27}
\]

In the case where the final kinetic energy is half the barrier height, we have \(mv_f^2 = U_0\) and we get

\[
v_i = \frac{v_f}{e^{-T/\tau}} \tag{29}
\]

This can be expressed in terms of the barrier length from \(22\):

\[
L = v_f \tau \left( 1 - e^{-T/\tau} \right) \tag{30}
\]

\[
v_i = \frac{v_f}{1 - L/v_f \tau} \tag{31}
\]

We’d like to find a set of values such that both \(v_i\) and \(v_f\) are positive, and also the initial kinetic energy is less than \(U_0\). Choosing \(L = v_f \tau / 4\) gives
Therefore the particle will actually tunnel through the barrier. However, given the crazy predictions arising from the reaction force, I’m not sure how much I would believe this result.