VELOCITY ADDITION IN SPECIAL RELATIVITY

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The velocity addition formula in special relativity is (if \( v_a \) and \( v_b \) are parallel):

\[
v_r = \frac{v_a + v_b}{1 + \frac{v_a v_b}{c^2}}
\]  

(1)

**Example 1.** To get a feel for how small the correction to the classical formula \( v_c = v_a + v_b \) is for everyday speeds, suppose \( v_a = 5 \text{ miles/hour} = 2.235 \text{ m s}^{-1} \) and \( v_b = 60 \text{ miles/hour} = 26.82 \text{ m s}^{-1} \). Since \( v_a v_b/c^2 \) in this case is very small, we can approximate \( v_r \) by

\[
v_r \approx (v_a + v_b) \left(1 - \frac{v_a v_b}{c^2}\right)
\]  

(2)

The percentage error in the classical formula is then

\[
\Delta v = \frac{v_c - v_r}{v_c} \times 100\%
\]  

(3)

\[
\approx \frac{(v_a + v_b) \left(\frac{v_a v_b}{c^2}\right)}{v_a + v_b} \times 100\%
\]  

(4)

\[
= \frac{v_a v_b}{c^2} \times 100\%
\]  

(5)

\[
= \frac{2.235 \times 26.82}{(3 \times 10^8)^2} \times 100\%
\]  

(6)

\[
= 6.66 \times 10^{-14}\%
\]  

(7)

It’s not surprising that no relativistic effects are seen in the everyday world.

**Example 2.** Suppose you could run at \( v_a = 0.5c \) (relative to the train) down the corridor of a train travelling at \( v_b = 0.75c \). An observer on the ground would see your speed relative to the ground as
\[
v = \frac{0.5 + 0.75}{1 + (0.5)(0.75)} c \]
\[
= 0.91 c
\]

(8)

(9)

Even though the classical sum of velocities is greater than \( c \), the relativistic formula still gives a result that is less than \( c \).

**Example 3.** The formula always gives a result that is less than \( c \). To prove this, we can simplify the notation by using velocities that are fractions of \( c \) so that \( a \equiv v_a / c \) and \( b \equiv v_b / c \) with the sum given by \( s \equiv v_r / c \). Then

\[
s = \frac{a + b}{1 + ab}
\]

(10)

To check if there are any extrema in the region \( 0 \leq a \leq 1, 0 \leq b \leq 1 \) we can take the two partial derivatives and set them to zero:

\[
\frac{\partial s}{\partial a} = \frac{1}{1 + ab} - \frac{(a + b)b}{(1 + ab)^2} = 0
\]

(11)

\[
\frac{\partial s}{\partial b} = \frac{1}{1 + ab} - \frac{(a + b)a}{(1 + ab)^2} = 0
\]

(12)

The only solution within the region is \( a = b = 1 \). We can see that this must be a maximum within the region, since along the border \( a = 1 \) or the border \( b = 1 \) we have \( s = 1 \), along the border \( a = 0 \) we have \( s = b \) and along the border \( b = 0 \) we have \( s = a \) so that \( s \leq 1 \) on all borders. Thus \( s \leq 1 \) everywhere in the region \( 0 \leq a \leq 1, 0 \leq b \leq 1 \). The surface [10] within this region looks like this:
The point $a = b = 1$ is, however, actually a saddle point, as an expanded plot of the region $0 \leq a \leq 2$, $0 \leq b \leq 2$ reveals:

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