INVARINCE OF SCALAR PRODUCT UNDER LORENTZ TRANSFORMATIONS

Although the time and position of an event can change under Lorentz transformations, the scalar product of any two four-vectors is an invariant under a Lorentz transformation. That is

$$\bar{a}_i \bar{b}^i = a_i b^i$$  \hspace{1cm} (1)

where for motion along the 1-axis ($x$ axis) the transformations are

$$\bar{a}^0 = \gamma (a^0 - \beta a^1)$$  \hspace{1cm} (2)
$$\bar{a}^1 = \gamma (a^1 - \beta a^0)$$  \hspace{1cm} (3)
$$\bar{a}^2 = a^2$$  \hspace{1cm} (4)
$$\bar{a}^3 = a^3$$  \hspace{1cm} (5)

and

$$a_0 = -a^0$$  \hspace{1cm} (6)
$$a_j = a^j$$  \hspace{1cm} (7)

for $j = 1, 2, 3$. We can see this directly by calculation, using $\gamma^2 = 1/\left(1 - \beta^2\right)$:
\[ \bar{a}_i b^i = -\gamma^2 (a^0 - \beta a^1) (b^0 - \beta b^1) + \gamma^2 (a^1 - \beta a^0) (b^1 - \beta b^0) + a^2 b^2 + a^3 b^3 \]  
\[ = \gamma^2 [a^0 b^0 (-1 + \beta^2) + a^1 b^0 (\beta - \beta) + a^0 b^1 (\beta - \beta) + a^1 b^1 (-\beta^2 + 1)] + a^2 b^2 + a^3 b^3 \]  
\[ = -a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3 \]  
\[ = a_i b^i \]  

In particular if \( a^i \) is the space-time four-vector of an event, or the difference between the four-vectors of two events, then

\[ \bar{a}_i \bar{a}^i = a_i a^i \]  

This leads to the invariant interval between two events:

\[ \Delta s^2 \equiv -\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 \]