An alternative way of writing the Lorentz transformations is to define a quantity called the \textit{rapidity}:

\[ \theta \equiv \tanh^{-1} \beta \] (1)

Using this definition, we have

\[ \gamma = \frac{1}{\sqrt{1 - \beta^2}} \] (2)

\[ = \frac{1}{\sqrt{1 - \tanh^2 \theta}} \] (3)

\[ = \frac{\cosh \theta}{\sqrt{\cosh^2 \theta - \sinh^2 \theta}} \] (4)

\[ = \cosh \theta \] (5)

since \( \cosh^2 \theta - \sinh^2 \theta = 1 \).

Also

\[ \gamma \beta = \cosh \theta \tanh \theta = \sinh \theta \] (6)

so

\[
\Lambda = \begin{bmatrix}
\gamma & -\beta \gamma & 0 & 0 \\
-\beta \gamma & \gamma & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
cosh \theta & -\sinh \theta & 0 & 0 \\
-\sinh \theta & \cosh \theta & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\] (7)
RAPIDITY

This is similar to a rotation through an angle $\theta$ in 3-d space, except both the sinh terms are negative.

The velocity addition formula becomes

\[ \bar{u} = \frac{u + v}{1 + uv/c^2} \]  
\[ = \frac{\beta_u + \beta_v c}{1 + \beta_u \beta_v c} \]  
\[ \beta \bar{u} = \frac{\tanh \theta_u + \tanh \theta_v}{1 + \tanh \theta_u \tanh \theta_v} \]  
\[ \tanh \theta \bar{u} = \tanh (\theta_u + \theta_v) \]

where in the last line we’ve used the formula for the tanh of a sum of two arguments.

The rapidities therefore simply add, giving a simpler measure of relativistic velocity:

\[ \theta \bar{u} = \theta_u + \theta_v \]

PINGBACKS

Pingback: Four-velocity again
Pingback: Generators of the Lorentz group for 4-vectors