FOUR-VELOCITY OF A PARTICLE IN HYPERBOLIC MOTION

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[Griffiths’s approach to the relativistic four-velocity is similar to that of Moore, although rather confusingly, he uses different notation (as well as keeping factors of $c$ in the equations rather than setting $c = 1$). To keep the notation consistent with Griffiths, I’ll use his notation here, but anyone attempting to follow both books should beware.]

We can now return to a particle travelling on a hyperbolic trajectory, so its position (one dimensional, on the $x$ axis) is

$$x(t) = \sqrt{b^2 + c^2 t^2} \quad (1)$$

Here, $x$ and $t$ are the position and time as measured by an observer at rest (so they are not proper time for the particle). We can find the particle’s proper time as a function of $t$ by using the relation between time intervals:

$$d\tau = \sqrt{1 - u^2/c^2} dt \quad (2)$$

We have

$$u = \dot{x} = \frac{c^2 t}{\sqrt{b^2 + c^2 t^2}} \quad (3)$$

$$d\tau = \sqrt{1 - \frac{c^2 t^2}{b^2 + c^2 t^2}} dt \quad (4)$$

$$= \frac{b}{\sqrt{b^2 + c^2 t^2}} dt \quad (5)$$

Integrating both sides, we get, taking $\tau = 0$ when $t = 0$ (the integral can be done by software or looked up as it is a standard integral):
\[ \tau = b \int_0^t \frac{1}{\sqrt{b^2 + c^2(t')^2}} dt' \quad (7) \]

\[ \tau = \frac{b}{c} \ln \left[ \frac{1}{b} \left( ct + \sqrt{b^2 + c^2 t^2} \right) \right] \quad (8) \]

We can write the position as a function of \( \tau \) by starting with (1) and using this last result:

\[ \tau = \frac{b}{c} \ln \left[ \frac{1}{b} (ct + x) \right] \quad (9) \]

\[ ct = \sqrt{x^2 - b^2} \quad (10) \]

\[ \tau = \frac{b}{c} \ln \left[ \frac{1}{b} \left( \sqrt{x^2 - b^2} + x \right) \right] \quad (11) \]

\[ be^{c\tau/b} = \sqrt{x^2 - b^2} + x \quad (12) \]

\[ \left( be^{c\tau/b} - x \right)^2 = x^2 - b^2 \quad (13) \]

\[ x = \frac{b}{2} \left( e^{c\tau/b} + e^{-c\tau/b} \right) \quad (14) \]

\[ = b \cosh \frac{c\tau}{b} \quad (15) \]

For the ordinary velocity, we have

\[ u = \frac{c^2 t}{\sqrt{b^2 + c^2 t^2}} \quad (16) \]

\[ = \frac{c \sqrt{x^2 - b^2}}{x} \quad (17) \]

\[ = cb \sqrt{\cosh^2 \frac{c\tau}{b} - 1} \frac{1}{b \cosh \frac{c\tau}{b}} \quad (18) \]

\[ = ctanh \frac{c\tau}{b} \]

The four velocity is defined as

\[ \eta^i = \frac{u^i}{\sqrt{1 - u^2/c^2}} \quad (19) \]

so we have (I’m assuming that in part (c) of Griffiths’s problem, he meant to
ask for $\eta^i$ in terms of $\tau$, not $t$, as the latter doesn’t give anything particularly informative):

\[
\eta^0 = \frac{c}{\sqrt{1 - \tanh^2 \frac{c\tau}{b}}} \quad (20)
\]
\[
= c \cosh \frac{c\tau}{b} \quad (21)
\]
\[
\eta_x = \frac{u}{\sqrt{1 - \tanh^2 \frac{c\tau}{b}}} \quad (22)
\]
\[
= \frac{c \tanh \frac{c\tau}{b}}{\sqrt{1 - \tanh^2 \frac{c\tau}{b}}} \quad (23)
\]
\[
= c \sinh \frac{c\tau}{b} \quad (24)
\]

As a check, we note that

\[
\eta_i \eta^i = -c^2 \left( \cosh^2 \frac{c\tau}{b} - \sinh^2 \frac{c\tau}{b} \right) = -c^2 \quad (25)
\]