

RELATIVISTIC TRANSFORMATION OF ELECTROMAGNETIC WAVES; THE DOPPLER EFFECT

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References: Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition; Pearson Education - Chapter 12, Problem 12.47.

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Let's apply the transformation equations for electromagnetic fields to an electromagnetic wave. Suppose we have a plane EM wave with frequency ω in the lab frame. The wave is polarized in the y direction and is travelling in the x direction, so its fields are

$$\mathbf{E} = E_0 e^{i(kx - \omega t)} \hat{\mathbf{y}} \quad (1)$$

$$\mathbf{B} = \frac{E_0}{c} e^{i(kx - \omega t)} \hat{\mathbf{z}} \quad (2)$$

where

$$k = \frac{\omega}{c} \quad (3)$$

To see how this wave looks in a frame moving with velocity v in the x direction, we can use the transformation equations

$$\bar{E}_y = \gamma(E_y - vB_z) \quad (4)$$

$$\bar{B}_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right) \quad (5)$$

We get

$$\bar{\mathbf{E}} = \gamma E_0 \left(1 - \frac{v}{c}\right) e^{i(kx - \omega t)} \hat{\mathbf{y}} \quad (6)$$

$$= E_0 \sqrt{\frac{1 - v/c}{1 + v/c}} e^{i(kx - \omega t)} \hat{\mathbf{y}} \quad (7)$$

$$\bar{\mathbf{B}} = \gamma \frac{E_0}{c} \left(1 - \frac{v}{c}\right) e^{i(kx - \omega t)} \hat{\mathbf{z}} \quad (8)$$

$$= \frac{E_0}{c} \sqrt{\frac{1 - v/c}{1 + v/c}} e^{i(kx - \omega t)} \hat{\mathbf{z}} \quad (9)$$

To get the final forms, we used

$$\gamma\left(1 - \frac{v}{c}\right) = \frac{1 - v/c}{\sqrt{1 - v^2/c^2}} \quad (10)$$

$$= \frac{1 - v/c}{\sqrt{(1 - v/c)(1 + v/c)}} \quad (11)$$

$$= \sqrt{\frac{1 - v/c}{1 + v/c}} \quad (12)$$

The amplitude of the wave gets smaller as the speed of the frame increases, becoming zero as $v \rightarrow c$.

To express this in the moving frame's coordinates, we use the (inverse) Lorentz transformations:

$$x = \gamma(\bar{x} + v\bar{t}) \quad (13)$$

$$t = \gamma\left(\bar{t} + \frac{v\bar{x}}{c^2}\right) \quad (14)$$

giving

$$kx - \omega t = \gamma\left(k - \frac{\omega v}{c^2}\right)\bar{x} - \gamma(\omega - kv)\bar{t} \quad (15)$$

$$= \gamma\frac{\omega}{c}\left(1 - \frac{v}{c}\right)\bar{x} - \gamma\omega\left(1 - \frac{v}{c}\right)\bar{t} \quad (16)$$

$$= \frac{\omega}{c}\sqrt{\frac{1 - v/c}{1 + v/c}}\bar{x} - \omega\sqrt{\frac{1 - v/c}{1 + v/c}}\bar{t} \quad (17)$$

Thus in the moving frame, the frequency of the wave is

$$\bar{\omega} = \omega\sqrt{\frac{1 - v/c}{1 + v/c}} \quad (18)$$

and the wavelength is

$$\bar{\lambda} = \frac{2\pi}{\bar{k}} \quad (19)$$

$$= \frac{2\pi c}{\bar{\omega}} \quad (20)$$

$$= \frac{2\pi c}{\omega} \sqrt{\frac{1+v/c}{1-v/c}} \quad (21)$$

$$= \lambda \sqrt{\frac{1+v/c}{1-v/c}} \quad (22)$$

That is, the wavelength gets longer, approaching infinity as $v \rightarrow c$, while the frequency gets smaller, approaching zero as $v \rightarrow c$. This is the Doppler effect for light. If $v > 0$ so that we are moving in the direction of propagation of the wave, the wavelength gets longer resulting in a red-shift. If $v < 0$ so that we are moving against the direction of propagation, the wavelength gets shorter and we have a blue-shift.

However, note that the speed of the wave is

$$\bar{c} = \frac{\bar{\lambda}\bar{\omega}}{2\pi} = c \quad (23)$$

so not surprisingly, the speed of the wave remains the same in the moving frame.

The intensity of the wave in the lab frame is given by

$$I = \frac{E_0^2}{2} c \epsilon_0 \quad (24)$$

In the moving frame this becomes

$$\bar{I} = \frac{E_0^2}{2} c \epsilon_0 \frac{1-v/c}{1+v/c} \quad (25)$$

so the ratio is

$$\frac{\bar{I}}{I} = \frac{1-v/c}{1+v/c} \quad (26)$$

and the intensity drops to zero as $v \rightarrow c$.

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