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The Minkowski force $K$ is the rate of change of four-momentum with respect to proper time, and allows Newton’s law to be written in its natural form

$$K = m\alpha$$

(1)

where $\alpha$ is the proper acceleration, or second derivative of position with respect to proper time. Here we’ll investigate the behaviour of a particle subject to a constant Minkowski force in one dimension.

In terms of ordinary force, we have

$$K = \frac{dp}{d\tau} = \frac{dp}{dt} \frac{dt}{d\tau} = \frac{1}{\sqrt{1-u^2/c^2}} F$$

(2)

The ordinary momentum $p$ is

$$p = \frac{mu}{\sqrt{1-u^2/c^2}}$$

(3)

so its derivative is

$$\frac{dp}{dt} = \frac{m}{\sqrt{1-u^2/c^2}} \frac{du}{dt} + \frac{mu^2}{(1-u^2/c^2)^{3/2}} \frac{du}{dt}$$

(4)

Inserting this into (2) we get

$$\frac{K}{m} dt = \frac{du}{1-u^2/c^2} + \frac{u^2 du}{(1-u^2/c^2)^2}$$

(5)

We can integrate both sides (using software, or integral tables) to get

$$\frac{K}{m} t + C = \frac{c}{4} \ln \left[ \frac{c+u}{c-u} \right] + \frac{c^2}{4} \left[ \frac{1}{c-u} - \frac{1}{c+u} \right]$$

(6)
where \( C \) is a constant of integration. If the initial conditions are \( u = 0 \) at \( t = 0 \), then \( C = 0 \) and we have

\[
\frac{K}{m} t = \frac{c}{4} \ln \left[ \frac{c + u}{c - u} \right] + \frac{c^2}{4} \left[ \frac{1}{c - u} - \frac{1}{c + u} \right]
\]  

(7)

This is an implicit equation for the speed of the particle as a function of time. If we want the position as a function of time, we need a relation between \( u \) and \( x \). Returning to \( 2 \) and \( 3 \) we have

\[
\sqrt{1 - u^2/c^2} \frac{K}{m} = \frac{d}{dx} \left( \frac{u}{\sqrt{1 - u^2/c^2}} \right)
\]  

(8)

We can use the chain rule to convert the derivative on the RHS to a derivative with respect to \( x \) by multiplying both sides by \( dt/dx \)

\[
\frac{dt}{dx} \sqrt{1 - u^2/c^2} \frac{K}{m} = \frac{d}{dx} \frac{u}{\sqrt{1 - u^2/c^2}} = \frac{d}{dx} \left( \frac{u}{\sqrt{1 - u^2/c^2}} \right)
\]  

(9)

Now \( dx/dt = u \) so \( dt/dx = 1/u \) and

\[
\sqrt{1 - u^2/c^2} \frac{K}{m} \frac{u}{K/m} = \frac{d}{dx} \left( \frac{u}{\sqrt{1 - u^2/c^2}} \right)
\]  

(10)

If we call the expression in the parentheses on the RHS \( A \), then we can integrate with respect to \( x \) (since \( K/m \) is a constant):

\[
A \equiv \frac{u}{\sqrt{1 - u^2/c^2}}
\]  

(11)

\[
\frac{1}{A} m \frac{K}{x + C} = \frac{dA}{dx}
\]  

(12)

\[
\frac{K}{m} x + C = \frac{1}{2} A^2
\]  

(13)

Again, starting from rest at the origin we have \( u = 0 \) when \( x = 0 \) so \( A = 0 \) also, and therefore \( C = 0 \), so we have

\[
A = \frac{u}{\sqrt{1 - u^2/c^2}} = \sqrt{\frac{2Kx}{m}}
\]  

(14)

At this point we could get a relation between \( x \) and \( t \) by solving \( 14 \) for \( u \) in terms of \( x \) and then substituting this into \( 7 \). For reference, we get
\[ u = \sqrt{\frac{2Kx}{m}} \frac{1}{\sqrt{1 + \frac{2Kx}{mc^2}}} \]  

(15)

so substituting will give something of a mess. To get the answer given in Griffiths requires a bit of algebra, but here is how I did it. Griffiths defines the quantity \( z \) as

\[ z \equiv \sqrt{\frac{2Kx}{mc^2}} \]  

(16)

\[ = \frac{A}{c} \]  

(17)

\[ = \frac{u}{c\sqrt{1-u^2/c^2}} \]  

(18)

The quantities appearing in Griffiths’s answer are

\[ \sqrt{1+z^2} = \frac{c}{\sqrt{c^2-u^2}} \]  

(19)

\[ z\sqrt{1+z^2} = \frac{u}{c(1-u^2/c^2)} \]  

(20)

We can rewrite (7) to get

\[ \frac{2Kt}{mc} = \frac{1}{2} \ln \left[ \frac{c+u}{c-u} \right] + \frac{c}{2} \left[ \frac{1}{c-u} - \frac{1}{c+u} \right] \]  

(21)

We’ll deal with the logarithm first. Its argument is

\[ \frac{c+u}{c-u} = \frac{(c+u)^2}{c^2 (1-u^2/c^2)} \]  

(22)

\[ = \frac{2u}{c(1-u^2/c^2)} + \frac{u^2 + c^2}{c^2 - u^2} \]  

(23)

\[ = \frac{2u}{c(1-u^2/c^2)} + \frac{c^2 - u^2 + 2u^2}{c^2 - u^2} \]  

(24)

\[ = \frac{2u}{c(1-u^2/c^2)} + 1 + \frac{2u^2}{c^2 (1-u^2/c^2)} \]  

(25)

Now we also have
\[(z + \sqrt{1 + z^2})^2 = 2z^2 + 2z\sqrt{1 + z^2} + 1 \quad (26)\]
\[= \frac{2u^2}{c^2(1 - u^2/c^2)} + \frac{2u}{c(1 - u^2/c^2)} + 1 \quad (27)\]
\[= \frac{c + u}{c - u} \quad (28)\]

Therefore
\[
\frac{1}{2} \ln \left[ \frac{c + u}{c - u} \right] = \ln \sqrt{\frac{c + u}{c - u}} \quad (29)
\]
\[= \ln \left( z + \sqrt{1 + z^2} \right) \quad (30)\]

For the second term in (21) we have
\[
c \left[ \frac{1}{c - u} - \frac{1}{c + u} \right] = \frac{c}{2} \frac{2u}{c^2(1 - u^2/c^2)} \quad (31)
\]
\[= \frac{u}{c(1 - u^2/c^2)} \quad (32)\]
\[= z\sqrt{1 + z^2} \quad (33)\]

Putting it all together, we have
\[
\frac{2Kt}{mc} = \ln \left( z + \sqrt{1 + z^2} \right) + z\sqrt{1 + z^2} \quad (34)\]